Empirical Mass Relations Involving the Strongly Interacting Particles*

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A set of mass relations which connect the masses of the strongly interacting particles is presented. All of the observed particle masses can be represented with good accuracy by a simple additive relation, in which there enter only two fundamental constants. We have also obtained many examples of the relation $m_3=m_1+m_2$ where m_1, m_2, and m_3 are the masses of three strongly interacting particles. It is shown that there exist many cases of particle pairs having the same mass difference, i.e., $m_1-m_2=m_3-m_4$. Moreover, there are about 20 sequences of particles, where a sequence is defined as a group of three or more particles with the same mass spacing. A set of simple mass relations has been obtained for the K meson, and the $K^*(725)$ and $K^*(888)$ resonances. The relation of the mass of the μ meson to the mass spectrum of the strongly interacting particles has been investigated. Finally, it has been found that for a large number of particle pairs (a,b), we have the relation $m_b=\lambda m_a$, where λ is a simple fraction.

I. INTRODUCTION

THE purpose of this paper is to point out a set of mass relations involving the mesonic resonances and the baryon isobars. This work can be regarded as an extension of previous results on empirical mass relations involving the baryon isobars.[1–3] All of the mass relations considered in the present work are linear in the masses.[4] As an example, we will consider a number of relations of the form: $m_3=m_1+m_2$, and $m_1-m_2=m_3-m_4$, where m_i ($i=1, 2, 3, 4$) are the masses of strongly interacting particles. The relation $m_3=m_1+m_2$ has the same form as that given by a production threshold relation, or else it can be interpreted as meaning that particle 3 is a compound of particles 1 and 2, in the sense of a loosely bound nucleus, with very small binding energy. However, it should be noted that whereas $m_3=m_1+m_2$ can be interpreted in some cases in terms of a threshold or compound nucleus effect,[5] if the isotopic spin I, baryon number B, and strangeness S of particles 1, 2, and 3 are such that 3 can be regarded as a compound of 1 and 2, we will also note examples of this mass relation where the existence of 3 can no longer be interpreted in this manner, i.e., when the quantum numbers I, B, and S of particles 1, 2, and 3 do not have the appropriate relations for 3 to be a compound of 1 and 2.

For those mass relations which cannot be interpreted in terms of a compound model, there is, of course, always the possibility that they may represent merely numerology. However, some of the mass relations are very striking by their simplicity and symmetry, and they are presented in this paper in the hope that they are physically meaningful, and that they may eventually contribute to our understanding of the mechanism of the strong interactions. We will not give in this paper all of the linear mass relations which have been obtained, but essentially only those which seem to be particularly simple, as examples of the various types of mass relations.[6]

In Sec. II, it will be shown that with a suitable definition of the pion mass m_π, and using the constant κ introduced by Takabayasi and Ohnuki,[7] the masses m of all of the strongly interacting particles can be expressed in the form:

$$m=pm_\pi+q\kappa\,, \tag{1}$$

where p and q are integers. In particular, for the sequence of nucleon isobars whose ground state is the well-known $\frac{3}{2}$, $\frac{3}{2}$ resonance,[1,3] namely $N_{3/2}^*(1238)$, $N_{1/2}^*(1512)$, $N_{3/2}^*(1922)$, and $N_{1/2}^*(2197)$, the mass can be expressed simply as: $m=pm_\pi$ (i.e., $q=0$), where $p=9$, 11, 14, and 16, respectively, for the four states involved. The fit of Eq. (1) to the data is generally within 4 MeV. Throughout this work, we have used the experimental mass values given by Rosenfeld,[8] wherever they are available. The representation of the masses according to Eq. (1) also brings up a number of interesting features of the mass differences, especially when these are just a multiple of m_π or κ. It has been shown that the probability that a random mass distribution would reproduce the actual agreement of the observed masses with Eq. (1) is essentially negligible (see the Appendix).

In Sec. III, we will point out a number of linear mass relations which involve only the masses of the observed particles, i.e., which do not involve the constant κ. In

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[1] T. F. Kycia and K. F. Riley, Phys. Rev. Letters **10**, 266 (1963).
[2] R. M. Sternheimer, Phys. Rev. Letters **10**, 309 (1963).
[3] R. M. Sternheimer, Phys. Rev. **131**, 2698 (1963).
[4] A preliminary account of the present work has been published in a letter by R. M. Sternheimer [Phys. Rev. Letters **13**, 37 (1964)].
[5] The fact that for a few baryon resonances, the mass corresponds to a threshold for two-particle production has been known for some time. See, e.g., S. F. Tuan, Nuovo Cimento **23**, 448 (1962).
[6] A more detailed account of the present work is given in Brookhaven National Laboratory Report BNL-8123 (unpublished). Copies of this report can be obtained from the author. See also R. M. Sternheimer, Phys. Rev. Letters **13**, 358 (1964).
[7] T. Takabayasi and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **30**, 272 (1963).
[8] A. H. Rosenfeld, in *Proceedings of the 1962 Conference on High Energy Physics at CERN, Geneva*, edited by J. Prentki (CERN Scientific Information Service, Geneva, 1962), p. 783.

particular, we will show that the relation $m_3=m_1+m_2$ holds in many cases in which particles 1 and 2 could not combine to form particle 3, in the manner of a quasinucleus, as described in Refs. 1 and 3. It appears that the number of cases for which $m_3=m_1+m_2$ is considerably larger than would be expected from a random mass distribution, and therefore such linear relations may turn out to be physically meaningful, even though they are not based on a compound model. In this connection, it may be pointed out that the mass relations mentioned above for the nucleon isobars, e.g., $m[N_{3/2}^*(1238)]=9m_\pi$, cannot be understood in terms of a quasinucleus model, since a loosely bound aggregate of nine pions would not have baryon number 1, and half-integral spin and isotopic spin, as does the isobar $N_{3/2}^*(1238)$.

Among the other mass relations which we will discuss in Sec. III are the following: (1) $m_1-m_2=m_3-m_4$, i.e., equal mass differences; (2) $m_3=m_1+\frac{1}{2}m_2$, of which there exist several examples[9]; (3) mass relations involving the μ meson mass; (4) $m_2=\lambda m_1$, where λ is a simple fraction, i.e., a rational number with small numerator and denominator. In particular, we have investigated the cases where λ is half-integral ($\lambda=\frac{3}{2}, \frac{5}{2}, \frac{7}{2}$), and quarter integral ($\lambda=3/4, 5/4, 7/4, 9/4$). There exist also several sets of three particles, such that $m_3=\lambda m_2$ and $m_2=\lambda m_1$, with the same value of λ, so that m_3 can be written as $m_3=\lambda^2 m_1$.

II. MASS RELATIONS INVOLVING m_π AND κ

In this section, we will show that the mass formula, Eq. (1), represents the masses of all of the presently known strongly interacting particles to a very good accuracy (see Tables I–III). Actually, it would be more pertinent to state that Eq. (1) follows in a very natural manner from some empirical observations on the mass spectrum (in particular, equal mass differences), so that Eq. (1) can be believed more strongly than if it were merely an arbitrary parametrization of the experimental mass values.

Throughout this paper, we will denote by m_π the average $\frac{1}{2}(m_{\pi^\pm}+m_{\pi^0})$ of the masses of $\pi^\pm$ and π^0; thus $m_\pi=137.3$ MeV. With this definition of m_π, we have[10]: $m_\eta=4m_\pi$ to within the small uncertainty of the experimental determination of the mass of the η meson: $m_{\eta,\mathrm{exp}}=549\pm2$ MeV.

We note that to a very good approximation, the mass of the well-known $N_{3/2}^*$ isobar with $J=I=\frac{3}{2}$ can be written as

$$m(N_{3/2}^*)=9m_\pi=1236\text{ MeV}. \quad (2)$$

The experimental value of $m(N_{3/2}^*)$ is 1238 ± 2 MeV. Here and in the following, the experimental values of the nucleon isobar masses are taken from Rosenfeld[8]; the experimental uncertainties of these mass values are those given in Ref. 1. According to Kycia and Riley,[1] the mass differences

$$m[N_{1/2}^*(1512)]-m[N_{3/2}^*(1238)]$$

and

$$m[N_{1/2}^*(2190)]-m[N_{3/2}^*(1920)]$$

are equal to $m_{\pi^+}+m_{\pi^0}$, which can be written as $2m_\pi$. Furthermore, the mass differences

$$m[N_{3/2}^*(1920)]-m[N_{3/2}^*(1238)]$$

and

$$m[N_{1/2}^*(2190)]-m[N_{1/2}^*(1512)]$$

are equal to $m_\eta+m_\pi$, which can be written as $5m_\pi$. Upon making use of these results, and of Eq. (2), we obtain for the three mass values involved:

$$m(N_{1/2}^*,\ 1512\pm2)=11m_\pi\ (=1510\text{ MeV})\,, \quad (3)$$

$$m(N_{3/2}^*,\ 1920\pm15)=14m_\pi\ (=1922\text{ MeV})\,, \quad (4)$$

$$m(N_{1/2}^*,\ 2190\pm20)=16m_\pi\ (=2197\text{ MeV})\,. \quad (5)$$

Here we have identified the states by giving the isotopic spin I (N_I^*) and the experimental value of the mass[8] with its associated uncertainty as estimated in Ref. 1. It can be concluded from Eqs. (2)–(5) that the masses of the four states which belong to the isobar system whose ground state is $N_{3/2}^*(1238)$ are just multiples of m_π.

We now consider the results of Takabayasi and Ohnuki,[7] who have shown that the masses of the $I=0$ mesons with strangeness $S=0$ are equally spaced,[11] at intervals of κ, where $\kappa\approx235$ MeV. Thus for the $I=0$ mesons,

$$m_n(I=0)=m_\eta+n\kappa\,, \quad (6)$$

where $n=0$ for the η meson, $n=1$ for ω, $n=2$ for φ, and $n=3$ for f. Since $m_\eta=4m_\pi$, Eq. (6) can be rewritten as follows:

$$m_n(I=0)=4m_\pi+n\kappa\,. \quad (7)$$

In order to obtain an equation for the mass m_ρ of the ρ meson, we make use of an empirical relation previously obtained by Sternheimer [Ref. 3, Eq. (11)], namely:

$$m_\rho+m_\omega=m_f+2m_\pi\,. \quad (8)$$

Upon inserting the expressions of Eq. (6) for m_ω and m_f, one obtains[12]:

$$m_\rho=2m_\pi+2\kappa\,. \quad (9)$$

It has been noted by Kycia[13] that the mass difference m_B-m_ρ is equal to 2κ, where m_B is the mass of the

[9] T. Takabayasi, Nuovo Cimento **30**, 1500 (1963).

[10] R. F. Peierls and S. B. Treiman, Phys. Rev. Letters **8**, 339 (1962).

[11] We remark that the relation $m_f-m_\varphi=m_\varphi-m_\omega$ was noticed independently by the present author. Results similar to those of Ref. 7 were also obtained by R. Kumar (private communication).

[12] This equation for m_ρ was also proposed by Takabayasi [see Ref. 9, Eq. (28)].

[13] T. F. Kycia (private communication).

recently discovered B particle[14] ($m_B \approx 1220$ MeV). Thus, for the $I=1$ mesons, we may have the equation:

$$m_n(I=1)=2m_\pi+2n\kappa\,, \tag{10}$$

where $n=1$ for ρ and $n=2$ for B. We also note that for $n=0$, one obtains $m_0=2m_\pi$. A quasiparticle having this mass ($2m_\pi=m_{\pi^+}+m_{\pi^0}$) and isotopic spin $I=1$ has been used by Kycia and Riley[1] and by Sternheimer[3] in their classification of the baryon isobars.

We note that to a very good accuracy, the mass of the $K^*(888)$ meson is given by

$$m[K^*(888)]=m_\rho+m_\pi\,. \tag{11}$$

With $m_\rho=750$ MeV, the right-hand side of Eq. (11) is 887 MeV. The reason for proposing Eq. (11) will be given below [see Eq. (66)]. Similarly to Eqs. (2)–(5), Eq. (11) cannot correspond to a compound model, since the system of a ρ meson and a pion would have strangeness $S=0$, and $I=0$, 1, or 2. Upon combining (11) with (9), we obtain

$$m[K^*(888)]=3m_\pi+2\kappa\,. \tag{12}$$

If we regard the hyperon isobar $Y_0^*(1815)$ as the combination of a nucleon and a $\bar{K}^*(888)$ particle,[2,4] we may write for its mass:

$$m[Y_0^*(1815)]=m_N+3m_\pi+2\kappa\,. \tag{13}$$

It has been noted by Takabayasi[9] that the value of κ is approximately given by $m_N/4$. Following this author, we define κ as $m_N/4=234.7$ MeV, where $m_N=\frac{1}{2}(m_n+m_p)$. Thus Eq. (13) becomes

$$m[Y_0^*(1815)]=3m_\pi+6\kappa\,. \tag{14}$$

Takabayasi[9] has also pointed out that the difference $m[Y_0^*(1815)]-m_\Lambda$ is closely given by 3κ, and that $m[Y_0^*(1405)]\approx 6\kappa$. Upon using these results and the mass differences shown in Fig. 1 of Ref. 3, one obtains the following expressions for the various Y_0^* and Y_1^* states and for the Λ hyperon:

$$m_\Lambda=m[Y_0^*(1815)]-3\kappa=3m_\pi+3\kappa \tag{15}$$

$$m[Y_1^*(1385)]=m_\Lambda+2m_\pi=5m_\pi+3\kappa\,, \tag{16}$$

$$m[Y_0^*(1520)]=m[Y_1^*(1385)]+m_\pi=6m_\pi+3\kappa\,, \tag{17}$$

$$m[Y_1^*(1660)]=m[Y_0^*(1520)]+m_\pi=7m_\pi+3\kappa\,. \tag{18}$$

We also obtain the following expression for the mass of the ABC particle (see Fig. 1 of Ref. 3):

$$m_{\rm ABC}=m[Y_0^*(1815)]-m[Y_0^*(1520)]=3\kappa-3m_\pi\,. \tag{19}$$

The right-hand side gives $m_{\rm ABC}=292$ MeV which is compatible with the recent experimental determination of Booth and Abashian,[15] according to which $m_{\rm ABC}$ is less than $2m_\pi+20$ MeV≈ 295 MeV.

[14] M. Abolins, R. L. Lander, W. A. Mehlhop, N. H. Xuong, and P. M. Yager, Phys. Rev. Letters 11, 381 (1963).

[15] N. E. Booth and A. Abashian, Phys. Rev. 132, 2314 (1963).

Upon using the Kycia-Riley scheme for the nucleon isobars $N_{3/2}^*(1650\pm25)$, $N_{1/2}^*(1688\pm3)$, and $N_{3/2}^*(2360\pm25)$, together with Eq. (9) for m_ρ, one obtains

$$m[N_{3/2}^*(1650\pm25)]=m_N+5m_\pi=5m_\pi+4\kappa\,, \tag{20}$$

$$m[N_{1/2}^*(1688\pm3)]=m_N+m_\rho=2m_\pi+6\kappa\,, \tag{21}$$

$$m[N_{3/2}^*(2360\pm25)]=m[N_{1/2}^*(1688)]+5m_\pi =7m_\pi+6\kappa\,. \tag{22}$$

We note that $N_{3/2}^*(1650\pm25)$ [Eq. (20)] corresponds to the "shoulder" of the π^+-p cross section.[16]

In the preceding equations [Eqs. (2)–(22)], we have obtained expressions for all of the strongly interacting particles, except for the following: K, $K^*(725)$, Σ, Ξ, $\Xi_{1/2}^*(1532)$, Ω, and three recently discovered resonances which will be discussed below.

In order to obtain an expression for m_K, we note that to within 2 MeV, the following simple mass relation holds:

$$m[K^*(888)]-m_K=m_N-m_\eta\approx 391 \text{ MeV}\,. \tag{23}$$

This relation was obtained in the same manner as Eq. (11), by using a linear mass transformation recently discovered by Wick,[17] which will be discussed below. In Eq. (23) and in the following, we use for m_K the average $\frac{1}{2}(m_{K^\pm}+m_{K^0})=\frac{1}{2}[(493.9\pm0.2)+(497.8\pm0.6)]=495.9\pm0.6$ MeV. Equation (23) gives

$$m_K=m[K^*(888)]+m_\eta-m_N\,. \tag{24}$$

Upon using Eq. (12) for $m[K^*(888)]$, and the relations: $m_\eta=4m_\pi$, $m_N=4\kappa$, we obtain from Eq. (24)

$$m_K=7m_\pi-2\kappa\,. \tag{25}$$

In connection with the $K^*(725)$ particle, we note that:

$$m[K^*(725)]-m_K\approx 229 \text{ MeV}\approx\kappa\,, \tag{26}$$

so that we can write:

$$m[K^*(725)]\approx 7m_\pi-\kappa\,. \tag{27}$$

In order to obtain an expression for the mass of the Σ hyperon, we note that we have the mass relation.[2,3]

$$m_\Sigma=m_N+m_\rho-m_K\,. \tag{28}$$

Upon inserting Eq. (9) for m_ρ and Eq. (25) for m_K into Eq. (28), we obtain

$$m_\Sigma=8\kappa-5m_\pi=2m_N-5m_\pi\,. \tag{29}$$

It has been pointed out by Takabayasi[9] that the ratio κ/m_π is very close to 12/7. Thus, an alternative expression for m_Σ is obtained by adding $12m_\pi-7\kappa$ to Eq. (29), which gives

$$m_\Sigma=7m_\pi+\kappa\,. \tag{30}$$

[16] P. Carruthers, Phys. Rev. Letters 4, 303 (1960).

[17] G. C. Wick (private communication).

For the experimental value of m_Σ, we will use the average mass[18]:

$$m_\Sigma=\tfrac{1}{3}[m(\Sigma^+)+m(\Sigma^0)+m(\Sigma^-)]$$
$$=1193.4\pm0.3\text{ MeV}. \quad (31)$$

We will now obtain expressions for the masses of the Ξ and Ω particles, and the $\Xi_{1/2}^*(1532)$ resonance. For this purpose, we note the following empirical relations:

$$m_\rho+m_\omega=m[\Xi_{1/2}^*(1532)], \quad (32)$$

$$m[Y_0^*(1815)]-m_\pi=m_\Omega. \quad (33)$$

Upon using $m_\rho=750$ MeV and $m_\omega=782$ MeV,[8] the left-hand side of (32) gives 1532 MeV, in very good agreement with the mass of the $\Xi_{1/2}^*$ state. The left-hand side of (33) is 1678 MeV. Concerning the mass of the Ω^- particle,[19] in view of the equal spacing of the $N_{3/2}^*(1238)$, $Y_1^*(1385)$, and $\Xi_{1/2}^*(1532)$ states,[20] one expects a mass of ≈1679 MeV, with which Eq. (33) is in good agreement.

In analogy to Eq. (32), we have the following similar mass relation for the $Y_1^*(1385)$ member of the SU_3 decuplet[20]:

$$m_K+m[K^*(888)]=m[Y_1^*(1385)]. \quad (34)$$

Upon using $m_K=496$ MeV, the left-hand side of (34) becomes 1384 MeV, in very good agreement with the mass of the Y_1^* state.

Upon inserting the expression for m_ρ [Eq. (9)] and m_ω [Eq. (7)] into Eq. (32), one obtains

$$m[\Xi_{1/2}^*(1532)]=6m_\pi+3\kappa. \quad (35)$$

In a similar fashion, from Eq. (33), one obtains by means of Eq. (14)

$$m_\Omega=2m_\pi+6\kappa. \quad (36)$$

It may be noted that Eq. (35) for $m[\Xi_{1/2}^*(1532)]$ is the same as the expression (17) for $m[Y_0^*(1520)]$. This result is at first sight disconcerting, since it means that the same mass formula applies to two baryon states, with different strangeness and different parity [assuming that $Y_0^*(1520)$ is a $D_{3/2}$ state]. This brings up the general question of the relation of the mass values to the other quantum numbers of the particle states (i.e., I, J, B, and S). This problem will be discussed in detail below. From the point of view of the agreement with experiment, we have $6m_\pi+3\kappa=1527.9$ MeV. This agrees to ~4 MeV with $m[\Xi_{1/2}^*(1532)]$, but for $m[Y_0^*(1520)]$, the discrepancy is ~8 MeV. Actually, the isobar $Y_0^*(1520)$ represents the case for which the deviation from the experimental value of the mass is largest (see Tables I–III).

[18] W. H. Barkas, J. N. Dyer, and H. H. Heckman, Phys. Rev. Letters **11**, 26 (1963).
[19] V. E. Barnes, P. L. Connolly, D. Crennell, B. Culwick, W. Delaney *et al.*, Phys. Rev. Letters **12**, 204 (1964).
[20] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

Finally, in order to obtain an expression for the mass of the Ξ particle, we note that we have the relation

$$m_\Xi+m_\varphi+m_\pi=2m[N_{3/2}^*(1238)]. \quad (37)$$

This mass relation, which may seem strange at first, was obtained by looking for relations that are similar to Eq. (29), which can be rewritten as follows:

$$m_\Sigma+m_\eta+m_\pi=2m_N. \quad (38)$$

With $m_\Xi=1319$ MeV [estimated average of $m(\Xi^-)$ and $m(\Xi^0)$] and $m_\varphi=1019$ MeV, Eq. (37) holds to 1 MeV. Upon inserting Eq. (2) for $m[N_{3/2}^*(1238)]$ and Eq. (7) for m_φ into Eq. (37), one obtains

$$m_\Xi+4m_\pi+2\kappa+m_\pi=18m_\pi, \quad (39)$$

whence

$$m_\Xi=13m_\pi-2\kappa. \quad (40)$$

The calculated value of the right-hand side is 1315.5 MeV, which differs by only ~3 MeV from the estimated average m_Ξ.

Equations (25) and (40) for m_K and m_Ξ, respectively, suggest that

$$m_\Xi-m_K=6m_\pi. \quad (41)$$

Upon using $m_\Xi\approx1319$ MeV, $m_K=496$ MeV, the left-hand side of (41) becomes 823 MeV, in very good agreement with $6m_\pi=823.8$ MeV. Thus the agreement of Eq. (41), which involves only observed masses, is actually closer than the agreement of the expressions for m_K and m_Ξ separately [Eqs. (25) and (40)].

All of the mass relations given above are of the following general form:

$$m=pm_\pi+q\kappa, \quad (42)$$

where p and q are integers (which are negative in some of the cases). Thus p and q may be in the nature of quantum numbers pertaining to the mass formula. For the $I=0$, $S=0$ mesons, we have $p=4$, and q is the same as the quantum number n used in Eq. (7). Similarly, for the $I=1$, $S=0$ mesons, we have $p=2$ and $q=2n$ [cf. Eq. (10)]. We note that Eq. (42) is analogous to that for the energy of an anisotropic harmonic oscillator in two dimensions, except for the fact that p or q becomes negative for some of the particles.

For the purpose of a direct comparison with experiment, Tables I–III give the experimental mass values and the calculated values from Eq. (42). The doublet of values (p,q) which pertains to each state is listed in the second column of each table. Table I pertains to the mesons, Table II includes the nucleon isobars, and Table III includes the hyperons and hyperon isobars. The estimated uncertainties of the experimental mass values have also been indicated. It is seen that the differences between calculated and experimental values are generally less than the experimental errors and in no case does the difference exceed 8 MeV. (In fact, in 20 out of 27 cases listed in Tables I–III, where a precise comparison is possible, the difference is $\leqq4$ MeV.) For

TABLE I. Mass values for the mesonic resonances. (All values are in MeV.)

Meson	(p,q)	m_{calc}	m_{exp}
η	(4,0)	549.2	549 ± 2
ω	(4,1)	783.9	782 ± 2
φ	(4,2)	1018.6	1019 ± 1
f	(4,3)	1253.3	1255 ± 5
ABC	$(-3, 3)$	292.2	~290
ρ	(2,2)	744.0	750 ± 5
B	(2,4)	1213.4	1220 ± 10
X	(7,0)	961.1	960 ± 5
K	$(7, -2)$	491.7	495.9 ± 0.6
$K^*(725)$	$(7, -1)$	726.4	725 ± 5
$K^*(888)$	(3,2)	881.3	888 ± 3
$K^*(1175)$	(0,5)	1173.5	1175 ± 5

TABLE II. Mass values for the nucleon isobars. The experimental uncertainties are those given in Ref. 1. (All values are in MeV.)

Isobar (m_{exp})	(p,q)	m_{calc}
$N(938.8)$	(0,4)	938.8
$N_{3/2}^*(1238\pm2)$	(9,0)	1235.7
$N_{1/2}^*(1485\pm5)$	(4,4)	1488.0
$N_{1/2}^*(1512\pm2)$	(11,0)	1510.3
$N_{3/2}^*(1650\pm25)$	(5,4)	1625.4
$N_{1/2}^*(1688\pm3)$	(2,6)	1682.8
$N_{3/2}^*(1920\pm15)$	(14,0)	1922.2
$N_{1/2}^*(2190\pm20)$	(16,0)	2196.8
$N_{3/2}^*(2360\pm25)$	(7,6)	2369.8

TABLE III. Mass values for the hyperons and hyperon isobars. (All values are in MeV.)

Particle (m_{exp})	(p,q)	m_{calc}
$\Lambda(1115.4\pm0.15)$	(3,3)	1116.0
$\Sigma(1193.4\pm0.3)$	(7,1)	1195.8
$Y_1^*(1385\pm5)$	(5,3)	1390.6
$Y_0^*(1405\pm5)$	(0,6)	1408.2
$Y_0^*(1520\pm5)$	(6,3)	1527.9
$Y_1^*(1660\pm10)$	(7,3)	1665.2
$Y_0^*(1815\pm20)$	(3,6)	1820.1
$\Xi(1319\pm2)$	$(13, -2)$	1315.5
$\Xi_{1/2}^*(1532\pm3)$	(6,3)	1527.9
$\Omega^-(1680\pm10)$	(2,6)	1682.8

some cases, the agreement is remarkably close. We mention in particular the nucleon isobars $N_{3/2}^*(1238)$, $N_{1/2}^*(1512)$, and $N_{3/2}^*(1922)$ and the Λ particle ($m_{\Lambda,\text{exp}}=1115.36\pm0.14$ MeV, $m_{\Lambda,\text{calc}}=1116.0$ MeV).

Before proceeding to a discussion of Eq. (42), we will point out some additional mass relations. We have noted the following empirical relations:

$$m_K+m[K^*(725)]=m_B\,, \qquad (43)$$

$$m_B+2\kappa=m[N_{1/2}^*(1688)]\,. \qquad (44)$$

Upon inserting Eqs. (25) and (27) into (42), we obtain

$$m_B=14m_\pi-3\kappa\,, \qquad (45)$$

so that Eq. (44) gives

$$m[N_{1/2}^*(1688)]=14m_\pi-\kappa=m[N_{3/2}^*(1922)]-\kappa\,, \qquad (46)$$

where the last step in Eq. (46) follows from Eq. (4) for $m[N_{3/2}^*(1922)]$.

The expressions (45) and (46) give very close agreement for m_B and $m[N_{1/2}^*(1688)]$. Thus the right-hand side of (45) equals 1218.1 MeV (as compared to $m_B=1220$ MeV). For Eq. (46), $14m_\pi-\kappa=1687.5$ MeV, in very good agreement with the experimental value.

From Eqs. (2), (45) and from Eq. (7) for m_f (with $n=3$), we obtain

$$\tfrac{1}{2}(m_f+m_B)=m[N_{3/2}^*(1238)]\,, \qquad (47)$$

which is well satisfied by the experimental mass values m_f and m_B. Thus with $m_f=1255$ MeV, $m_B=1220$ MeV, the value of the left-hand side of (47) is 1237.5 MeV.

An interesting feature of Eq. (47) is that the particles on the left side have baryon number $B=0$, whereas, of course, the isobar has $B=1$. This is another example of the fact that in the present scheme the masses of the mesons and baryons are interrelated.

After the work leading to Eq. (42) had been completed, three new resonances were discovered. All three mass values can be represented accurately by means of Eq. (42).

(1) The P_{11} resonance in the π^-p system corresponding to a mass $m=1485$ MeV, which was found by Roper,[21] corresponds to the state with $p=4$, $q=4$, for which Eq. (42) gives: $m=4m_\pi+4\kappa=1488$ MeV, in good agreement with the experimental value. A resonance state with $p=4$, $q=4$ was actually anticipated for two reasons: (a) the states with $p=q$ seem to be very important, as discussed below; thus (2,2) corresponds to the ρ meson; (3,3) represents the Λ particle, so that it was natural to expect that (4,4) might correspond to an observable resonance; (b) referring to Eq. (7), the states $n=0$, 1, 2, and 3 correspond to mesons with $I=0$, $S=0$. It was therefore expected that the state $n=4$ (with mass $m\approx1488$ MeV) might occur, and correspond to a meson with the same quantum numbers ($I=0$, $S=0$) as the states $n=0$, 1, 2, and 3. Instead, it appears that there actually exists a state with $n=4$, but it is a baryon isobar (with $I=\frac{1}{2}$, $S=0$). A similar situation exists concerning Eq. (10). The cases $n=1$ and $n=2$ correspond to mesons having $I=1$ and $S=0$. The state $n=3$ for which Eq. (10) gives a mass

$$m_3=2m_\pi+6\kappa\approx m_B+2\kappa \qquad (48)$$

corresponds to the $N_{1/2}^*(1688)$ nucleon isobar, [cf. Eq. (44)].

(2) The recently discovered meson[22,23] X^0 with mass $m\approx960$ MeV which decays into $\eta+2\pi$ corresponds to $p=7$, $q=0$ in Eq. (42), i.e., $m_X=7m_\pi$. We have $7m_\pi=961.1$ MeV, in very good agreement with the experimental value. We note that, similarly to Eqs.

[21] L. D. Roper, Phys. Rev. Letters 12, 340 (1964).

[22] G. R. Kalbfleisch, L. Alvarez, A. Barbaro-Galtieri, O. Dahl *et al.*, Phys. Rev. Letters 12, 527 (1964).

[23] M. Goldberg, M. Gundzik, S. Lichtman, J. Leitner, M. Primer *et al.*, Phys. Rev. Letters 12, 546 (1964).

(7) and (10), we now have a sequence of levels with constant separation $\Delta m = 2m_\pi$, namely $X = (7,0)$; $N_{3/2}^*(1238) = (9,0)$, $N_{1/2}^*(1512) = (11,0)$. Here and in the following, we denote the particles by their (p,q) assignment. Again we encounter the feature that a meson and two baryons are combined in the same sequence. Incidentally, if we accept the quasiparticles $(\eta\pi) = (5,0)$, and $(3\pi) = (3,0)$ used in the compound particle model of Refs. 1 and 3, then we have an extension of the previous sequence to include the pion $[=(1,0)]$, (3π), $(\eta\pi)$, X, $N_{3/2}^*(1238)$, and $N_{1/2}^*(1512)$, with 5 equal mass spacings $\Delta m = 2m_\pi$.

In connection with the X meson, the Q value for the decay into $\eta + 2\pi$ is given by

$$Q(X) = m_X - m_\eta - 2m_\pi = 7m_\pi - 4m_\pi - 2m_\pi = m_\pi. \quad (49)$$

It may be noted that for the $\eta \to 3\pi$ decay, the Q value is also m_π. In general, since the mass values are represented by Eq. (42), the Q values for the decay of the particles will also have the form of Eq. (42), except possibly for some of the leptonic decays. The relation of the muon mass to Eq. (42) will be discussed below.

(3) A group at Wisconsin[24] has found a resonant state with a mass $m \approx 1175$ MeV, which decays into $K + 2\pi$. We note that the mass value is very closely given by $5\kappa = 1173.5$ MeV. Moreover, if this resonance is confirmed, it would serve to form a sequence together with the nucleon $(m_N = 4\kappa)$ and the $Y_0^*(1405)$ state $(m = 6\kappa)$. This sequence has the same mass spacing $\Delta m = \kappa$ as the $I=0$, $S=0$ meson sequence: η, ω, φ, and f [Eq. (7)].

In view of the fact that in the present scheme, the mass of a particle is characterized by the two integers p and q, it seems reasonable to inquire whether the values of p and q give some information about the other properties of the particle, i.e., its isotopic spin I, angular momentum J, parity P, strangeness and baryon number. However, an inspection of the (p,q) assignments shows that no such correlation of p and q with the other properties is directly apparent. In fact, there exists at least one, and very probably, two counterexamples, in which two particles with the same quantum numbers I, J, B, S have different masses: (1) the first case is the well-known ω,φ pair; both particles have the J^P quantum numbers 1^- (with G parity $= -1$), and their masses differ by κ; (2) the second case is the pair consisting of the nucleon and the P_{11} resonance of Roper[21] at $m = 1485$ MeV. Both of these are $P_{1/2}$ states, with isotopic spin $I = \frac{1}{2}$, their masses differ by $4m_\pi$, i.e., by the mass of an η meson. In fact, in this particular case, the compound model of Refs. 1 and 3 can be used, namely the P_{11} isobar can be regarded as the combination of a nucleon plus an η meson, i.e., (N,η). Such a combination would have strangeness $S=0$ and isotopic spin $I = \frac{1}{2}$, in agreement with the quantum numbers of the isobar.

Aside from these two very obvious counterexamples to the expectation that p and q might be unique functions of I, J, B, and S, we have several other similar situations. As an example, the masses of the nucleon isobars $N_{3/2}^*(1238)$, $N_{1/2}^*(1512)$, $N_{3/2}^*(1922)$, and $N_{1/2}^*(2197)$ are particularly simple, since they are just multiples of m_π, i.e., $q=0$ in Eq. (42). Moreover, the quantum numbers B and S are, of course, the same for these four states ($B=1$, $S=0$). One might hope that isotopic spin $I=\frac{1}{2}$ would be associated with even (or odd) p, while $I=\frac{3}{2}$ would be associated with odd (or even) p. However, the values of p for $N_{3/2}^*(1238)$ and $N_{1/2}^*(1512)$ are both odd, but the I values are different. The same situation exists for $N_{3/2}^*(1922)$ and $N_{1/2}^*(2197)$, where both p values are even. Actually, one can derive a complicated dependence of p on I and J which will just fit these four states. However, the resulting equation for $p(I,J)$ looks very artificial and would not fit the p values for the nucleon and the other nucleon isobars. Perhaps these results for the lack of a simple dependence of p on the other quantum numbers are not so surprising, when one considers that $N_{3/2}^*(1238)$ and $N_{1/2}^*(1512)$ are part of the same sequence which includes the 960-MeV resonance (X meson), as discussed above. Thus there is a change in baryon number between the X particle and the $N_{3/2}^*(1238)$ isobar, and this change could not have been predicted by just considering the value of p.

The fact that there is no simple correlation of p and q with J and P (parity) is also shown by the assignments $Y_1^*(1385) = (5,3) = P_{3/2}$ state; $Y_0^*(1520) = (6,3) = D_{3/2}$; $Y_1^*(1660) = (7,3) = D_{3/2}$. It has not been possible to obtain a formula for p in terms of J, I, and L (orbital angular momentum) which would fit these three states and also give agreement for the Λ particle $[=(3,3)]$.

It has been suggested by Peierls[25] that it might be of

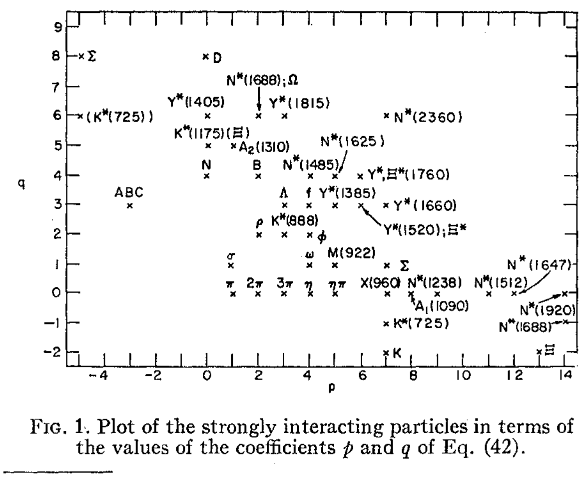

FIG. 1. Plot of the strongly interacting particles in terms of the values of the coefficients p and q of Eq. (42).

[24] T. P. Wangler, A. R. Erwin, and W. D. Walker, Phys. Letters 9, 71 (1964).

[25] R. F. Peierls (private communication).

interest to plot the assignments (p,q) for the various particle states on a graph of p versus q. This plot is shown in Fig. 1. For the three particles Ξ, Σ, and $K^*(725)$, two alternative (p,q) assignments have been shown. In the figure, we have also included the quasi-particles 2π, 3π, and $\eta\pi$, which have been used in the compound model for the classification of isobars, as given in Refs. 1–3. The $N_{1/2}^*(2197)$ isobar with $p=16$, $q=0$ would lie outside the limits of the figure, to the right of $N_{3/2}^*(1920)$.

In addition to the particles discussed above, this figure also includes the deuteron D ($p=0$, $q=8$), and the following resonance states which have been reported or confirmed after the main part of this paper was completed: (1) the σ meson[26] ($\sigma\rightarrow\pi^++\pi^-$) with mass $m\sim380$ MeV ($p=q=1$); (2) the two resonances in the $\pi\rho$ system,[27] $A_1(1090)$ ($p=8$, $q=0$), and $A_2(1310)$ ($p=1$, $q=5$); (3) a possible resonance[28] in the reaction $\pi^-+p\rightarrow\Lambda+K^0$, with mass $m=1647$ MeV$=12m_\pi$ ($p=12$, $q=0$); (4) two states with mass $m\approx1760$ MeV, $Y^*(1760)$[29] and $\Xi^*(1760)$[30] for which the assignment is $p=6$, $q=4$; (5) a possible $S=0$ meson[31] decaying into $\pi^++\pi^-$, with $m=922\pm30$ MeV ($p=5$, $q=1$). It is noteworthy that all of the 8 observed particle states for which $m=pm_\pi$ ($p>1$), i.e., $q=0$, have strangeness $S=0$. These states are: $\eta[p=4]$, $X(960)$ [7], $A_1(1090)$ [8], $N_{3/2}^*(1238)$ [9], $N_{1/2}^*(1512)$ [11], $N_{1/2}^*(1647)$ [12], $N_{3/2}^*(1922)$ [14], and $N_{1/2}^*(2197)$ [16], where the number in the square brackets gives the value of p in Eq. (42). Since the number of particles with $S=0$ represents about 60% of all particles, the probability that a random distribution of states would give rise to the observed correlation with $S=0$ is $(0.6)^8=0.017$.

Figure 1 shows that there exist 18 sequences each consisting of three or more particles, separated by a constant mass interval. The longest sequences (each having five particles) are as follows: (1) the (η,ω,φ,f) sequence (interval$=\kappa$), discussed in Ref. 7, to which we can add the $N_{1/2}^*(1485)$ state; (2) a sequence going from Λ to $Y_1^*(1660)$, with interval m_π; (3) the sequence (π, 2π, 3π, η, $\eta\pi$) with interval m_π; (4) the sequence [φ, Λ, B, A_2, $Y_0^*(1405)$] with interval $\kappa-m_\pi$ ($=97$ MeV). There are also two sequences with interval 2κ, namely: [$K^*(725)$, Σ, $Y_1^*(1660)$] and [2π, ρ, B, $N_{1/2}^*(1688)$]. Some of the preceding sequences appear to mix up the baryons and mesons. Thus, in the sequence extending from the Λ to $Y_1^*(1660)$, the f meson comes between the Λ and the $Y_1^*(1385)$. With the recently discovered X meson ($m=960$ MeV), one also obtains a new sequence of four particles with $\Delta m=\kappa$, namely K, $K^*(725)$, $X(960)$, and Σ.

From Fig. 1, one can deduce directly a number of mass relations, in particular those for which a mass difference m_a-m_b equals a multiple of m_π, or κ, or $\kappa-m_\pi$. Among the additional relations which can be deduced from Fig. 1 or from the (p,q) assignments, we list the following:

$$m_\Xi-m_K=m_\Lambda-m_{\mathrm{ABC}}=m_X-m_\pi=6m_\pi\,, \tag{50}$$

$$\begin{aligned}m_\Sigma-m_K&=m[N_{3/2}^*(2360)]-m[Y_1^*(1660)]\\&=m[Y_1^*(1660)]-m_X\\&=m[Y_0^*(1815)]-m_\Lambda\approx3\kappa\,,\end{aligned} \tag{51}$$

$$m[N_{3/2}^*(1920)]=2m_X\,, \tag{52}$$

$$m[N_{1/2}^*(1688)]-m_B=m[Y_0^*(1405)]-m_N=2\kappa\,. \tag{53}$$

In particular, Eq. (50) holds to within 2 MeV. Thus, $m_\Xi-m_K=1319-496=823$ MeV, $m_\Lambda-m_{\mathrm{ABC}}=1115-290=825$ MeV, $m_X-m_\pi=960-137=823$ MeV, as compared to $6m_\pi=823.8$ MeV. The other relations have essentially the same accuracy as (50), if one uses the experimental mass values, except that for Eq. (51), the mass differences are ~700 MeV, as compared to $3\kappa=704.1$ MeV.

Among the various states (p,q), those for which either p or q is the square of an integer, and the other index is 0, appear to be especially important. Thus (4,0) corresponds to the η meson, which is the lowest $I=0$ meson state (aside from the ABC). (9,0) is the $N_{3/2}^*(1238)$ resonance which forms the ground state of an isobar system.[1] (16,0) is the $N_{1/2}^*(2197)$ resonance which forms a link between the two nucleon isobar systems of Kycia and Riley.[1] Finally, (0,4) corresponds to the nucleon. (0,9) gives a mass $9\kappa=2112$ MeV, which is somewhat above the mass region in which the existence of resonances with $S=-1$ or $S=-2$ has been thoroughly investigated.

The states for which $p=q$ also appear to have some special importance. Thus, (2,2) corresponds to the ρ meson, which is the lowest mass $I=1$ meson (above the pion). Similarly (3,3) corresponds to the Λ hyperon, which can be regarded as the ground state of the entire system of $S=-1$ isobars, as shown in Fig. 1 of Ref. 3. The state (4,4) which had been anticipated from the present arguments has been recently found as a resonance in the π^-p system,[21] as discussed above. The doublet (1,1) corresponds to the σ meson, for which additional evidence has been recently obtained.[26]

In addition to the particles shown in Fig. 1, we also wish to discuss the following states: (1) The mesonic resonance decaying into $K_1^0K^\pm\pi^\mp$ which was first ob-

[26] R. Del Fabbro, M. De Pretis, R. Jones, G. Marini, A. Odian, G. Stoppini, and L. Tau, Phys. Rev. Letters **12**, 674 (1964). Evidence for the σ meson has also been obtained by F. S. Crawford, R. A. Grossman, L. J. Lloyd, L. R. Price, and E. C. Fowler, *ibid.* **11**, 564 (1963), and by J. Kirz, J. Schwartz, and R. D. Tripp, Phys. Rev. **130**, 2481 (1963). See also L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962).

[27] S. U. Chung, O. Dahl, L. Hardy, R. Hess, G. Kalbfleisch *et al.*, Phys. Rev. Letters **12**, 621 (1964); M. Aderholz *et al.*, Phys. Letters **10**, 226 (1964).

[28] G. T. Hoff, Phys. Rev. Letters **12**, 652 (1964); L. Bertanza, P. Connolly, B. Culwick, F. Eisler, T. Morris *et al.*, *ibid.* **8**, 332 (1962).

[29] A. Barbaro-Galtieri, A. Hussain, and R. D. Tripp, Phys. Letters **6**, 296 (1963).

[30] P. Bellière *et al.*, Phys. Letters **6**, 316 (1963); A. Halsteinslid *et al.*, *Proceedings of the Sienna International Conference on Elementary Particles (1963)*, Vol. 1, p. 173.

[31] H. Hulubei *et al.*, Phys. Letters **6**, 77 (1963).

served by Armenteros *et al.*[32] has a mass $m=1.41$ BeV, which is very close to $6\kappa=1408$ MeV. Hence the (p,q) assignment would be (0,6), which is the same as that for the $Y_0^*(1405)$ isobar. Thus we have a case where a meson and a baryon have the same (p,q) values. (2) A similar situation exists at $m\approx 2360$ MeV if the resonance state[33] decaying into $\Lambda+N$ is confirmed. This mass value is essentially the same as that of the $N_{3/2}^*(2360\pm 25)$ nucleon isobar. Thus the (Λ,N) state would have the same (p,q) doublet as $N_{3/2}^*(2360)$, namely (7,6). (3) The two recently discovered resonances in the π^+p and π^-p total cross sections[34] satisfy several interesting mass relations. The mass values are $m(N_{1/2}^*)=2645\pm 10$ MeV and $m(N_{3/2}^*)=2825\pm 15$ MeV for the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ states, respectively. The mass difference $2825-2645=180$ MeV is essentially the same as that between the two lower states,[35] $N_{3/2}^*(2360\pm 25)$ and $N_{1/2}^*(2190\pm 20)$. Both mass differences are given by $3m_\pi-\kappa=177$ MeV. Moreover, $m[N_{3/2}^*(2645)]-m[N_{3/2}^*(2360)]\approx 2m_\pi$. The (p,q) assignments are therefore (9,6) for $N_{1/2}^*(2645)$ and (12,5) or (0,12) for $N_{3/2}^*(2825)$, where we have made use of $7\kappa\approx 12m_\pi$. Thus we have: $m[N_{3/2}^*(2825)]\approx 3m_N$, and from the point of view of the compound model, this state could therefore be regarded as a quasinucleus consisting of a nucleon plus a nucleon-antinucleon pair (in the $I=1$ state) with negligible binding energy.

It is of interest to write down the (p,q) assignments which correspond to the octets of the SU_3 scheme.[20]

(1) The particles of the scalar meson octet (π, K, η) are represented by

$$\pi=(1,0);\quad K=(7,-2);\quad \eta=(4,0). \tag{54}$$

(2) For the vector meson octet [ρ, $K^*(888)$, and ω or φ], we obtain

$$\rho=(2,2);\quad K^*(888)=(3,2);$$

and

$$\omega=(4,1)\quad\text{or}\quad \varphi=(4,2). \tag{55}$$

(3) For the baryon octet, we find

$$N=(0,4);\quad \Lambda=(3,3);\quad \Sigma=(7,1);\quad \Xi=(13,-2). \tag{56}$$

The only apparent regularity is that the three vector mesons ρ, $K^*(888)$, and φ have the same q and equally spaced p values. This leads to the relation

$$\tfrac{1}{2}(m_\rho+m_\varphi)=m[K^*(888)]. \tag{57}$$

With $m_\rho=750$ MeV, $m_\varphi=1019$ MeV, the left-hand side becomes 884.5 MeV as compared to Rosenfeld's value 888 MeV for m_{K^*}.

For the $J^P=\frac{3}{2}^+$ decuplet, namely $N_{3/2}^*(1238)$, $Y_1^*(1385)$, $\Xi_{1/2}^*(1532)$, and Ω, we have obtained the following assignments:

$$N_{3/2}^*(1238)=(9,0);\quad Y_1^*(1385)=(5,3);$$
$$\Xi_{1/2}^*(1532)=(6,3);\quad \Omega=(2,6). \tag{58}$$

Thus, we find

$$m(\Xi_{1/2}^*)-m(N_{3/2}^*)=m(\Omega)-m(Y_1^*)=3\kappa-3m_\pi=m_{\mathrm{ABC}}, \tag{59}$$

where we have used Eq. (19) for m_{ABC}.

The spacings between adjacent states (i.e., with S differing by one unit) are slightly different from one another, with the present assignments. However, the average spacing (i.e., one-half the spacing between two states with $\Delta S=2$) is $\frac{1}{2}m_{\mathrm{ABC}}=146$ MeV, which is closely equal to the experimental value ($\sim$147 MeV).

In connection with Eqs. (45) and (46), we note that an alternative expression for m_ρ can be obtained from Eq. (46), namely:

$$m_\rho=m[N_{1/2}^*(1688)]-m_N=14m_\pi-5\kappa, \tag{60}$$

where we have made use of the fact that the $N_{1/2}^*(1688)$ isobar corresponds to the threshold for the reaction $\pi+N\to N+\rho$. The relation (60) gives $m_\rho=748.7$ MeV, in very good agreement with the experimental value[8] $m_\rho=750$ MeV. We note that Eq. (60) corresponds also to adding $12m_\pi-7\kappa$ to the right hand side of Eq. (9) for m_ρ, making use of the approximate relation $\kappa/m_\pi\approx 12/7$. Equation (60) agrees somewhat better with the experimental value than Eq. (9) which gives 744.0 MeV (see Table I).

Upon inserting (11) into (60), we obtain the following alternate expression for $m[K^*(888)]$:

$$m[K^*(888)]=15m_\pi-5\kappa. \tag{61}$$

The right-hand side equals 886.0 MeV, in good agreement with the experimental value.

We will now discuss the method by which the mass relations (11) and (23) were obtained. Wick[17] has recently considered the dependence on the masses m_i of the allowed regions of the relativistic variables s, t, and u for the general reaction $1+2\to 3+4$. These regions are defined by the three branches of a cubic curve for which the equation was first derived by Kibble.[36] Wick[17] has shown that if the masses involved in a given reaction are denoted by m_i $(i=1,\cdots 4)$, then the same region of allowed s, t, and u values (i.e., the same cubic) will be obtained if the m_i are replaced by either of two sets of derived masses m_i' and m_i'' $(i=1,\cdots 4)$, where m_i' and m_i'' are given by two linear transformations of the original m_i values. As an

[32] R. Armenteros *et al.*, *Proceedings of the Sienna International Conference on Elementary Particles (1963)*, Vol. 1, p. 287. T. H. Tan, D. Miller, and J. Steinberger, Bull. Am. Phys. Soc. **9**, 459 (1964).
[33] P. A. Piroué, Phys. Letters **11**, 164 (1964).
[34] A. Citron, W. Galbraith, T. F. Kycia, B. A. Leontic, R. H. Phillips, and A. Rousset, Phys. Rev. Letters **13**, 205 (1964).
[35] A. N. Diddens, E. W. Jenkins, T. F. Kycia, and K. F. Riley, Phys. Rev. Letters **10**, 262 (1963).
[36] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

example, we list the expressions for m_3' and m_3'':

$$m_3'=\tfrac{1}{2}(-m_1-m_2+m_3+m_4)\,, \tag{62}$$

$$m_3''=\tfrac{1}{2}(m_1-m_2+m_3+m_4)\,. \tag{63}$$

In Eqs. (62) and (63), it is assumed that the masses m_i have been labeled as follows: $m_1\leqq m_2\leqq m_3\leqq m_4$. If it turns out that a derived mass (m_i' or m_i'') is equal to the mass of an observed particle m, or to $\frac{1}{2}m$, then one can obtain a new mass relation, as will be shown below.

We consider the reaction: $\pi+N\rightarrow\Sigma+K$. One obtains $m_1'=53$, $m_2'=307$, $m_3'=750$, $m_4'=1383$; $m_1''=190$, $m_2''=444$, $m_3''=887$, $m_4''=1246$ MeV. Thus, we have $m_3'=m_\rho$, $m_2''=\frac{1}{2}m[K^*(888)]$, $m_3''\approx m[K^*(888)]$. The other derived masses m_i' and m_i'' are uninteresting from the present point of view (except possibly $m_4'\approx m[Y_1^*(1385)]$).

From the expressions for m_3' and m_3'' [Eqs. (62) and (63)], we obtain the relations:

$$m_\rho=m_3'=\tfrac{1}{2}(-m_\pi-m_K+m_N+m_\Sigma)\,, \tag{64}$$

$$m[K^*(888)]\approx m_3''=\tfrac{1}{2}(m_\pi-m_K+m_N+m_\Sigma)\,. \tag{65}$$

Upon subtracting (64) from (65), we find:

$$m[K^*(888)]-m_\rho=m_\pi\,, \tag{66}$$

which is equivalent to Eq. (11).

Upon adding (64) and (65), we find

$$\begin{aligned} m_\rho+m[K^*(888)]&=m_N+m_\Sigma-m_K\\ &=2m_N+m_\rho-2m_K\,, \end{aligned} \tag{67}$$

where we have used Eq. (28) for m_Σ. One finally obtains

$$m[K^*(888)]=2m_N-2m_K\,, \tag{68}$$

which can be written as follows:

$$m_K+\tfrac{1}{2}m[K^*(888)]=m_N\,. \tag{69}$$

This mass relation is satisfied to within 1 MeV (with $m_K=496$, $m_N=939$ MeV). Equation (69) can be regarded as the analog for the strangeness $S=\pm1$ mesons of a similar relation proposed by Takabayasi,[9] namely:

$$m_\eta+\tfrac{1}{2}m_\omega=m_N\,. \tag{70}$$

In comparing (69) and (70), we note that according to the SU_3 scheme, K and η belong to the same octet, in the same manner as K^* and ω.

It can be shown that the use of $m_2''=\frac{1}{2}m[K^*(888)]$, together with Eqs. (64) and (65) would not lead to any additional results, besides the two mass relations given above, Eqs. (66) and (68). It may be noted that by applying the transformations $(m_i)\rightarrow(m_i')$ and $(m_i)\rightarrow(m_i'')$ to the reaction $\pi+N\rightarrow\Lambda+K$, one does not get any useful results.

Another reaction which has been found to lead to a new mass relation is: $\rho+\omega\rightarrow K+\bar{K}^*(888)$. One finds that $m_4''=961$ MeV, which equals $\frac{1}{2}\times1922$ MeV, where 1922 MeV is the mass of the $I=\frac{3}{2}$, $F_{7/2}$ nucleon isobar. One obtains from m_4''

$$m[N_{3/2}^*(1922)]=2m_4''=-m_K+m_\rho+m_\omega+m_{K^*}\,. \tag{71}$$

Here and in the following discussion, $m_{K^*}\equiv m[K^*(888)]$. We will use the following expression for the mass of the $I=\frac{3}{2}$, $J=\frac{7}{2}$ isobar:

$$m[N_{3/2}^*(1922)]=m_N+m_f-2m_\pi\,. \tag{72}$$

In order to derive Eq. (72), we note that the $I=\frac{1}{2}$ level at $\sim$2195 MeV can be written as either[1-3] m_N+m_f or $m[N_{3/2}^*(1922)]+2m_\pi$. From Eqs. (71) and (72), we find:

$$m_N+m_f-2m_\pi=-m_K+m_\rho+m_\omega+m_{K^*}\,. \tag{73}$$

Upon inserting Eq. (8) into (73) one obtains

$$m_N-4m_\pi=m_{K^*}-m_K\,. \tag{74}$$

Upon setting $4m_\pi=m_\eta$, one obtains Eq. (23).

III. MASS RELATIONS NOT INVOLVING κ

In this section, we will present several types of mass relations which have in common the feature that they do not involve directly the constant κ. In other words, these mass relations involve only the physically observable particles. They are on the whole more accurate than those of Eq. (42). For the additive relations of Parts (A)–(E), the accuracy is always to within 2 MeV, and generally to within 1 MeV.

All of the mass relations to be discussed below are of the following form:

$$n_1m_1+n_2m_2=n_3m_3+n_4m_4\,, \tag{75}$$

where n_1, n_2, n_3, and n_4 are integers. In most cases, $n_4=0$. Moreover, except for some of the cases for which $m_b=\lambda m_a$ (λ=fraction), the integers n_i are small ($\leqq3$). The following types of mass relations will be considered:

(A): $m_3=m_1+m_2$;

(B): $m_3=m_1+\frac{1}{2}m_2$;

(C): relations involving the K, $K^*(725)$, and $K^*(888)$ mesons;

(D) and (E): relations involving equal mass differences, i.e., $m_1-m_2=m_3-m_4$. These sections will include a discussion of the relation of the mass of the μ meson to the present scheme;

(F): multiplicative mass relations involving only two particles, i.e., $m_b=\lambda m_a$, where λ is a rational number, i.e., a simple fraction. In most of the cases, the denominator of λ is small. In terms of Eq. (75), we have $\lambda=n_1/n_3$, with $n_2=n_4=0$.

A. The Mass Relations $m_3=m_1+m_2$

Concerning the relations of the form

$$m_3=m_1+m_2\,, \tag{76}$$

we have given numerous examples of this type of relation in Sec. II. It should be noted that the mass relations of the compound-particle model[1–3] are of the form (76). However, as discussed in detail in Sec. II, this relation holds also in many cases where a compound of particles 1 and 2 could not have the same quantum numbers (I, J, B, S) as particle 3. Among the previous examples of (76), we may mention the following: Eqs. (11), (32), (33), (34), (43), (52), and (59).

A more systematic way of investigating Eq. (76) consists in obtaining all of the mass differences Δm among the 32 presently known strongly interacting particles, and in deriving the properties of the distribution $F(\Delta m)$ of the mass differences. We note that, as expected from Eq. (42), the distribution $F(\Delta m)$ seems to show definite clusters around certain particular values, with relatively large intervals between the clusters. In particular, there are clusters centered around the masses of known particles. In the following, we give the numbers of times that various particle masses occur in the spectrum of the Δm values: m_π: 7 times; m_{ABC}: 9 times; m_K: 5 times; m_η: 3 times; $m[K^*(725)]$: 5 times; m_ρ: 5 times; m_ω: once; $m[K^*(888)]$: 3 times; $m[X(960)]$: 6 times; m_φ: twice; m_f: 3 times, giving a total of 49 mass differences.

Aside from those mass differences which are essentially equal to the mass of an observed particle, there are about 50 mass differences which are equal to a multiple of m_π, or to κ or a multiple of κ.

It is easily shown that two particle pairs will have equal mass differences if on the (p,q) plot the four particles form a parallelogram. If the particles are labeled 1, 3, 2, 4, as we go around the parallelogram, e.g., clockwise, then we have

$$p_1+p_2=p_3+p_4\,, \tag{77}$$

$$q_1+q_2=q_3+q_4\,, \tag{78}$$

whence

$$m_1+m_2=m_3+m_4\,. \tag{79}$$

From Eq. (79), it follows that we have two cases of pairs with equal Δm, namely,

$$m_1-m_3=m_4-m_2\,, \tag{80}$$

and

$$m_1-m_4=m_3-m_2\,. \tag{81}$$

One can obtain several parallelograms from Fig. 1, e.g., $[\rho,\, f,\, \omega,\, 2\pi]$ corresponding to Eq. (8); $[N,\, N_{1/2}^*(1688),\, N_{3/2}^*(2360),\, N_{3/2}^*(1625)]$; $[\Lambda,\, Y_0^*(1520),\, X(960),\, \eta]$, $[Y_0^*(1405),\, Y_0^*(1815),\, N_{3/2}^*(1625),\, B]$; and $[K^*(725),\, Y_1^*(1385),\, X(960),\, \text{ABC}]$. We also have several rectangles, e.g., $[B,\, N_{1/2}^*(1485),\, \varphi,\, \rho]$; $[\rho,\, \varphi,\, \eta,\, 2\pi]$; and $[f,\, Y_1^*(1660),\, X(960),\, \eta]$.

B. The Mass Relations $m_3=m_1+\frac{1}{2}m_2$

We will now discuss the mass relations of the form

$$m_3=m_1+\tfrac{1}{2}m_2\,. \tag{82}$$

Two examples of this type of mass relation have been given above, namely Eqs. (69) and (70).

The relation (70) which has been noticed by Takabayasi[9] is important for one of his mass formulas [see Eq. (7) of Ref. 9]. Since we have obtained a very similar mass relation, namely Eq. (69), the question arises as to whether the mass of some of the other particles can also be expressed in the form of Eq. (82). We have not investigated this problem exhaustively. We have restricted ourselves to the mesons, the hyperons, and the $N_{3/2}^*(1238)$ isobar. In most cases, we have found that the mass can be expressed by the relation (82) very accurately. In fact, in many cases, there are several alternate expressions (82) for a given mass.

As an example, for the Σ hyperon, upon inserting Eq. (28) into Eq. (69), one obtains

$$m_\Sigma=m_\rho+\tfrac{1}{2}m[K^*(888)]\,, \tag{83}$$

which is completely similar to Eq. (69) for m_N, except that m_ρ replaces m_K.

It is, of course, impossible to decide at the present time whether these relations are only numerology, or whether they may be useful for an understanding of the strong interactions. However, this type of mass relation seems to occur in a number of cases. In particular, as will be shown below, there are eight mass differences which can be written as $\frac{1}{2}m_\omega=391$ MeV. In fact, the mass differences of Eq. (23) provide two examples, and we can write

$$m[K^*(888)]=m_K+\tfrac{1}{2}m_\omega\,. \tag{84}$$

It may also be noted that relations of the type (82) have been previously considered by Matumoto.[37]

We have obtained the following accurate mass relations:

$$\begin{aligned} m_\Lambda &= m[K^*(725)]+\tfrac{1}{2}m_\omega \\ &= m_K+\tfrac{1}{2}m[N_{3/2}^*(1238)]\,, \end{aligned} \tag{85}$$

$$m_\Sigma=m[K^*(725)]+\tfrac{1}{2}m_N\,, \tag{86}$$

$$\begin{aligned} m[N_{3/2}^*(1238)] &= m[K^*(725)]+\tfrac{1}{2}m_\varphi \\ &= m_K+\tfrac{1}{2}m[N_{1/2}^*(1485)] \\ &= m[X(960)]+\tfrac{1}{2}m_\eta \\ &= m_\pi+\tfrac{1}{2}m[N_{1/2}^*(2197)]\,, \end{aligned} \tag{87}$$

$$\begin{aligned} m_f &= m_\eta+\tfrac{1}{2}m[Y_0^*(1405)] \\ &= m_K+\tfrac{1}{2}m[Y_0^*(1520)] \\ &= m_{\text{ABC}}+\tfrac{1}{2}m[N_{3/2}^*(1922)]\,, \end{aligned} \tag{88}$$

$$m_B=m_\rho+\tfrac{1}{2}m_N\,. \tag{89}$$

The fact that there are four alternate ways of expressing the mass of the $N_{3/2}^*(1238)$ isobar seems to be rather remarkable. It should be pointed out that some of the relations (85)–(89) could have been derived directly

[37] K. Matumoto, Progr. Theoret. Phys. (Kyoto) **27**, 1079 (1962).

from the (p,q) assignments of the particles involved. However, in many cases, this is not possible, e.g., for all cases for which the p and q pertaining to the mass m_2 of Eq. (82) are not even numbers (for instance, for the cases with $\frac{1}{2}m_\omega$, since $q=1$ for the ω).

C. Mass Relations for K, $K^*(725)$, and $K^*(888)$

We have obtained some particularly simple mass relations for the K meson, and the $K^*(725)$ and $K^*(888)$ resonances. In order to derive the relation for m_K, we can use Eqs. (23) and (68), namely,

$$m[K^*(888)]-m_K=m_N-m_\eta\,, \tag{23}$$

$$m[K^*(888)]+2m_K=2m_N\,. \tag{68}$$

Upon subtracting (23) from (68), one obtains

$$3m_K=m_N+m_\eta\,, \tag{90}$$

which can be written as follows:

$$3m_K=4m_\pi+4\kappa=1488\text{ MeV}. \tag{91}$$

The relation (90) or (91) holds within $\sim$1 MeV for $3m_K$. It should also be noted that $4m_\pi+4\kappa$ corresponds very closely to the mass of the recently discovered $N_{1/2}^*(1485)$ resonance.[21]

In view of the particularly simple and accurate relation (90), we have investigated whether $3m[K^*(725)]$ and $3m[K^*(888)]$ can also be expressed in a similar manner. The results are as follows:

$$3m[K^*(725)]=m_N+m[N_{3/2}^*(1238)]=9m_\pi+4\kappa\,, \tag{92}$$

$$3m[K^*(888)]=m[Y_0^*(1405)]+m_f=4m_\pi+9\kappa\,. \tag{93}$$

The accuracy of Eqs. (92) and (93) is similar to that of (90).

We note that $3m_\Lambda$ can be written as follows:

$$3m_\Lambda=m[Y_0^*(1815)]+m[\Xi_{1/2}^*(1532)]=9m_\pi+9\kappa\,. \tag{94}$$

The fact that $3m_\Lambda=9m_\pi+9\kappa$ follows, of course, directly from Eq. (15).

The mass relations (91)–(94) lead to the following additional results:

(a) The relation

$$m_\Lambda=(9/4)m_K\,, \tag{95}$$

is satisfied within the experimental error of m_K. Thus with $m_\Lambda=1115.36\pm0.15$ MeV, Eq. (95) gives

$$m_K=\frac{1115.36\pm0.15}{(9/4)}=495.72\pm0.07\text{ MeV}\,, \tag{96}$$

which is in very good agreement with the experimental value: $m_K=495.9\pm0.6$ MeV, obtained as $\frac{1}{2}(m_{K^+}+m_{K^0})$, where $m_{K^+}=493.9\pm0.2$ MeV and $m_{K^0}=497.8\pm0.6$ MeV.

(b) The relation

$$m[K^*(725)]+m[K^*(888)]=m_\Lambda+m_K \tag{97}$$

is satisfied within 2 MeV. It should perhaps be pointed out that $m_\Lambda+m_K=1611$ MeV corresponds to the threshold for $\Lambda+K$ production in πN collisions. It was noted in Ref. 3 that this threshold does not lead to a resonance maximum in the πN system, whereas the threshold for $\Sigma+K$ production is associated with a nucleon isobar, the $N_{1/2}^*(1688)$ state. Instead of being associated with an isobar, it appears that the ΛK threshold corresponds to the sum $m[K^*(725)]+m[K^*(888)]$, even though the reaction $K^*(725)+\bar{K}^*(888)\to\Lambda+K$ is, of course, forbidden by conservation of I, J, and B.

D. Mass Relations Involving the Mass of the μ Meson

Although the μ meson is a weakly interacting particle, it is nevertheless of interest to determine whether its mass is connected in any simple way with the masses of the strongly interacting particles and, in particular, with the constants m_π and κ which enter into Eq. (42). Indeed, one finds that $m_\pi/m_\mu=13/10$ and $\kappa/m_\mu=20/9$ to a very high accuracy. These fractions may seem strange at first sight, and the denominators are rather high. However, as will be shown below [in part (F), Sec. III], for a large number of pairs of strongly interacting particles, we have the relation $m_2=\lambda m_1$, where λ is a simple fraction (rational number). Cases with $\lambda=10/9$, $20/9$, and $13/10$ will also be shown to exist for several pairs of strongly interacting particles.

The experimental μ meson mass is given as[38] $(206.765\pm0.003)m_e=105.652\pm0.002$ MeV. With $m_\pi=\frac{1}{2}(139.59+135.00)=137.295\pm0.05$ MeV, one obtains

$$\frac{m_\pi}{m_\mu}=\frac{137.295\pm0.05}{105.652\pm0.002}=1.29950\pm0.00047\,, \tag{98}$$

as compared to $13/10=1.3$. For κ, we will use the following accurate value:

$$\kappa\equiv\tfrac{1}{8}(m_p+m_n)=\tfrac{1}{8}(938.213+939.507\pm0.02)$$
$$=234.715\pm0.003\text{ MeV}. \tag{99}$$

Then we obtain

$$\frac{\kappa}{m_\mu}=\frac{234.715\pm0.003}{105.652\pm0.002}=2.2216\,, \tag{100}$$

as compared to $20/9=2.2222$.

If one uses the relations

$$20m_\mu=9\kappa\,, \tag{101}$$

$$13m_\mu=10m_\pi\,, \tag{102}$$

[38] D. P. Hutchinson, J. Menes, G. Shapiro, and A. M. Patlach, Phys. Rev. 131 1351 (1963).

one obtains upon dividing (101) by (102)

$$9\kappa/10m_\pi=20/13\,, \tag{103}$$

so that $\kappa/m_\pi=200/117=1.7094$.

If one would assume that the ratio κ/m_μ is actually 20/9, one would obtain for the muon mass:

$$m_\mu=(234.715\pm0.003)/(20/9) = 105.622\pm0.0014\text{ MeV}\,, \tag{104}$$

which differs by only 30 keV from the measured mass value. We also note that if one would assume that m_π/m_μ is exactly 13/10, the result for m_μ would be

$$m_\mu=(137.295\pm0.05)/1.3 = 105.612\pm0.038\text{ MeV}. \tag{105}$$

This value has a larger standard deviation than (104) on account of the larger uncertainty of m_π, as compared to that of κ. Actually, Eq. (105) agrees with the experimental mass value to about one standard deviation, as was expected from the result of Eq. (98).

We have also attempted to express the μ meson mass by means of Eq. (42), with appropriate values of p and q. It was found that the closest agreement would be obtained with $p=11$, $q=-6$ for which we have

$$m_\mu\approx 11m_\pi-6\kappa=102.1\text{ MeV}. \tag{106}$$

This result was found by considering all possible combinations for which $|p|$ and $|q|$ are $\leqq16$. Obviously, with much larger values of $|p|$ and $|q|$, one would be expected to improve the agreement somewhat. However, we have restricted ourselves to values of p and q which are of the same order as those used in Fig. 1.

The error for m_μ obtained with (106) is: $105.6-102.1=3.5$ MeV, or roughly a factor 100 larger than that obtained with Eq. (104). We shall consider Eq. (106) again in the following discussion.

In connection with the (p,q) assignment for m_μ [Eq. (106)], one is led to the question as to whether m_μ appears as a mass difference between states belonging to the meson or baryon isobar systems. Indeed, one finds for the baryon isobars, three mass differences which are very closely equal to m_μ, i.e., well within the experimental errors of the isobar mass values. Thus we note that

$$m[N_{3/2}^*(1922)]-m[Y_0^*(1815)]=107\text{ MeV}\approx m_\mu\,, \tag{107}$$

and

$$m[N_{1/2}^*(1512)]-m[Y_0^*(1405)]=107\text{ MeV}\approx m_\mu. \tag{108}$$

Furthermore, if we use for the "shoulder" in the π^+p cross section the value suggested by Kycia and Riley,[1] namely: $m\ (N_{3/2}^*)=m_N+m_\eta+m_\pi=m_N+5m_\pi=1625$ MeV, we obtain the additional relation:

$$m[N_{3/2}^*(1625)]-m[Y_0^*(1520)]=105\text{ MeV}\approx m_\mu. \tag{109}$$

The mass relations (107)–(109) have several common features as far as the quantum numbers I, J, S, and the parity P are concerned. In order to show this, we write down the (most likely) I, J, S, and P assignments for the various states involved:

$$\begin{aligned} N_{3/2}^*(1922)&: I=\tfrac{3}{2},\quad J^P=\tfrac{7}{2}+,\quad S=0;\\ Y_0^*(1815)&: I=0,\quad J^P=\tfrac{5}{2}+,\quad S=-1.\\ N_{3/2}^*(1625)&: I=\tfrac{3}{2},\quad J^P=\tfrac{5}{2}-,\quad S=0;\\ Y_0^*(1520)&: I=0,\quad J^P=\tfrac{3}{2}-,\quad S=-1.\\ N_{1/2}^*(1512)&: I=\tfrac{1}{2},\quad J^P=\tfrac{3}{2}-,\quad S=0;\\ Y_0^*(1405)&: I=0,\quad J^P=\tfrac{1}{2}-,\quad S=-1.\end{aligned}$$

In each case, in going from the nucleon isobar to the hyperon isobar ($\Delta S=-1$), J decreases by one, and the parity remains unchanged. Also, in all three cases, the isotopic spin of the Y^* is $I=0$.

We now consider Eq. (106), and note that it is completely consistent with Eq. (108), in the following sense. We have previously noted that $m[N_{1/2}^*(1512)]=11m_\pi$ [see Eq. (3)], and, according to Takabayasi,[9] $m[Y_0^*(1405)]=6\kappa$. Thus Eq. (108) can be rewritten as follows:

$$11m_\pi-6\kappa\approx m_\mu\,, \tag{108a}$$

which is identical with Eq. (106). The comparison of (106) and (108) illustrates the fact, previously mentioned, that the relations between the masses of observed particle states are in some cases more accurate than the values given by the $pm_\pi+q\kappa$ expression. In the present case, the difference of the experimental mass values in Eq. (108) is 107 MeV, in better agreement with $m_\mu=105.6$ MeV than the value 102.1 MeV derived from (106).

In connection with the fact that m_μ appears three times as a mass difference between isobar states, it is reasonable to inquire whether m_μ also plays a role in the meson spectrum. Taking a hint from the fact that in all three cases, m_μ represents the mass difference between isobars whose strangeness S differs by one unit, we are led to consider as a possibility a pair of mesonic states, one of which has $|S|=1$, i.e., a K^* or K, the other having $S=0$. One finds that to a very good accuracy:

$$m[K^*(888)]-m_\omega=m_\mu. \tag{110}$$

With $m_\omega=782$ MeV, (110) gives 106 MeV on the left-hand side, in very good agreement with m_μ.

In view of Eq. (110), we have calculated all of the values $m_K\pm m_\mu$, $m[K^*(725)]\pm m_\mu$, and $m[K^*(888)]\pm m_\mu$. This has given us several additional simple mass relations. In particular, one finds that

$$m[K^*(888)]+m_\mu=993\text{ MeV}\approx 2m_K\,, \tag{111}$$

from which it follows that

$$m_K-m_\mu=m[K^*(888)]-m_K=m_N-m_\eta \approx 390\text{ MeV}. \tag{112}$$

In obtaining Eq. (112), we have used Eq. (23) in connection with m_N-m_η. Equation (112) shows that one has three consecutive mass levels: m_μ, m_K, and $m[K^*(888)]$, for which the two mass spacings: m_K-m_μ and $m[K^*(888)]-m_K$ are equal.

We have also obtained the following relations:

$$m[K^*(725)]-m_\mu=\tfrac{1}{2}m[N_{3/2}^*(1238)]=619\text{ MeV}, \quad (113)$$

$$m[K^*(725)]+m_\mu=\tfrac{1}{2}m[Y_1^*(1660)]=830\text{ MeV}. \quad (114)$$

From (113) and (114), one can derive that

$$4m[K^*(725)]=m[N_{3/2}^*(1238)]+m[Y_1^*(1660)], \quad (115)$$

and

$$m[Y_1^*(1660)]-m[N_{3/2}^*(1238)]=4m_\mu. \quad (116)$$

In fact, if in (116) one uses the best experimental values, 1660 and 1238 MeV, one obtains on the left-hand side: 422 MeV, which is in very close agreement with $4m_\mu=4\times105.6=422.4$ MeV.

We have also considered the mass values $m_0\pm m_\mu$, where m_0 is the mass of a strangeness $S=0$ meson. Thus we have obtained the following results:

$$m_\eta=m_\mu+\tfrac{1}{2}m[K^*(888)]=m_K+\tfrac{1}{2}m_\mu, \quad (117)$$

$$m_B=m_\Lambda+m_\mu, \quad (118)$$

We note that all of the preceding equations involving m_μ, i.e., Eqs. (107)–(118), are satisfied to within 2 MeV, and generally to within 1 MeV. Equation (117) is of the same form as Eqs. (85)–(89), i.e., $m_3=m_1+\frac{1}{2}m_2$. An additional relation of this type gives the mass of the K meson

$$m_K=m_\mu+\tfrac{1}{2}m_\omega. \quad (119)$$

In Eq. (118), we can write for m_B [cf. Eq. (43)]:

$$m_B=m_K+m[K^*(725)]=m_\Lambda+m_\mu, \quad (120)$$

whence

$$m_K-m_\mu=m_\Lambda-m[K^*(725)]=390\text{ MeV}. \quad (121)$$

Referring to Eq. (112), $m_\Lambda-m[K^*(725)]$ represents the fourth example of a mass difference of 390 MeV. The cases with $\Delta m=390$ MeV will be discussed below in more detail [in Sec. III, part (E)].

Upon combining Eq. (110) with Eq. (11) for $m[K^*(888)]$ we obtain

$$m_\rho+m_\pi-m_\omega=m_\mu, \quad (122)$$

whence

$$m_\omega-m_\rho=m_\pi-m_\mu. \quad (123)$$

With $m_\omega=782$ MeV, $m_\rho=750$ MeV, the left-hand side of (123) is 32 MeV, while the right-hand side equals: $137.3-105.6=31.7$ MeV, in very good agreement. It may be noted that such good agreement would not have been obtained if we had used $m_{\pi^\pm}(=139.6\text{ MeV})$ or $m_{\pi^0}(=135.0\text{ MeV})$, instead of the average pion mass m_π.

We note that aside from $m_\omega-m_\rho$, there are three other Δm values which are essentially equal to $m_\pi-m_\mu$. These mass differences are as follows:

$$m_f-m_B\approx m[Y_0^*(1520)]-m[N_{1/2}^*(1485)]$$
$$\approx m[Y_1^*(1660)]-m[N_{3/2}^*(1625)]\approx m_\pi-m_\mu. \quad (124)$$

The fact that $m_f-m_B=m_\pi-m_\mu$ follows directly from Eq. (123), since we have: $m_f=m_\omega+2\kappa$ and $m_B=m_\rho+2\kappa$ [see Eqs. (7) and (10)]. The two other mass differences of Eq. (124) have the same expression as m_f-m_B, namely $\Delta m\approx 2m_\pi-\kappa$.

We have also found that a number of mass differences are equal to multiples of m_μ, i.e., $2m_\mu$, $3m_\mu$, or $4m_\mu$, in the same manner as for $n\kappa$ and nm_π.

Thus we have found the following mass relations involving $2m_\mu$:

$$m[Y_0^*(1405)]-m_\Sigma=m[\Xi_{1/2}^*(1532)]-m_\Xi\approx 2m_\mu$$
$$=211.2\text{ MeV} \quad (125)$$

and the following sequences of four particles with common mass interval $\Delta m=2m_\mu$:

$$m[Y_1^*(1385)]-m[K^*(1175)]$$
$$=m[K^*(1175)]-m[X(960)]=m[X(960)]$$
$$-m_\rho=2m_\mu, \quad (126)$$

$$m[Y_0^*(1520)]-m[A_2(1310)]$$
$$\approx m[A_2(1310)]-m[A_1(1090)]\approx m[A_1(1090)]$$
$$-m[K^*(888)]\approx 2m_\mu. \quad (127)$$

We note that the masses of corresponding particles in the two sequences differ by just one pion mass; e.g.,

$$m[K^*(888)]=m_\rho+m_\pi;$$
$$m[Y_0^*(1520)]=m[Y_1^*(1385)]+m_\pi.$$

For $\Delta m=3m_\mu$, we have the following relations:

$$m_f-m_N=m[N_{1/2}^*(1512)]-m_\Sigma\approx 3m_\mu$$
$$=316.8\text{ MeV}. \quad (128)$$

The second of the relations (128), which involves $N_{1/2}^*(1512)$, follows directly from Eqs. (108) and (125).

For $\Delta m=4m_\mu$, we have the two relations [cf. Eq. (116)]:

$$m_\Omega-m_f=m[Y_1^*(1660)]-m[N_{3/2}^*(1238)]=4m_\mu, \quad (129)$$

in addition to the $\Delta m=4m_\mu$ cases which can be deduced from Eqs. (126) and (127).

It appears from the preceding results [Eqs. (107)–(129)], that the muon plays essentially the same role as m_π and κ, in connection with the mass differences. Thus the quantities m_μ, $2m_\mu$, $3m_\mu$, and $4m_\mu$ each appear several times in the mass difference spectrum, in the same manner as the quantities nm_π and $n\kappa$ (n=integer). This result is, of course, very surprising, since the muon is known to have only weak interactions. However, at this point, it should be remarked that the constant κ which has been extensively used above does not correspond to the mass of a physically observed particle, so that the formulation of the mass values in terms of m_π and κ is also somewhat unexpected.

One can ask the question as to whether the particle masses can be expressed in terms of m_π and m_μ (instead of m_π and κ), by using the expression

$$m=am_\pi+bm_\mu\,, \tag{130}$$

where a and b are integers. It turns out that this is possible (with the restriction that $-4\leqq a\leqq 16$, $-3\leqq b\leqq 8$, analogous to the restrictions on p and q in Fig. 1). One has to make use of Eqs. (101) and (102) in the conversion from Eq. (42) to Eq. (130) for the individual particle masses. The deviations from the experimental mass values are about the same as for Eq. (42) [$\lesssim 6$ MeV]. However, aside from the many sequences which involve values of $\Delta m=nm_\pi$, which are the same as in Fig. 1, there exist essentially only two sequences which involve $\Delta m=nm_\mu$, i.e., Eqs. (126) and (127). Thus Eq. (130) appears to be rather artificial as compared to Eq. (42), which reveals the existence of no less than 6 sequences with $\Delta m=n\kappa$.

E. Particle Pairs With Mass Difference $\Delta m=390$ MeV

In this section, we will give a brief discussion of the cases where the mass difference m_1-m_2 equals $m_N-m_\eta=390$ MeV. There are altogether 8 mass differences which are essentially equal to 390 MeV ($=4\kappa-4m_\pi$). Four of these have been given above, in Eqs. (112) and (121), namely:

$$m_K-m_\mu=m[K^*(888)]-m_K=m_\Lambda-m[K^*(725)]=m_N-m_\eta\,. \tag{131}$$

The other four mass differences with values $\approx m_N-m_\eta$ are as follows:

$$\begin{aligned} m[N_{3/2}^*(1922)]&-m[\Xi_{1/2}^*(1532)]\\ &\cong m[K^*(1175)]-m_\omega\cong m[Y_0^*(1405)]-m_\varphi\\ &\cong m[N_{3/2}^*(1625)]-m[N_{3/2}^*(1238)]\\ &\cong 390\text{ MeV}\,. \end{aligned} \tag{132}$$

It is seen that the common mass difference represents also the Q value for the $K_{\mu2}$ decay.

The mass difference m_N-m_η equals $\frac{1}{2}m_\omega$, as shown by Takabaysi,[9] and therefore all eight relations (131), (132) can be written in the form of Eq. (82), e.g.,

$$m[N_{3/2}^*(1922)]=m[\Xi_{1/2}^*(1532)]+\tfrac{1}{2}m_\omega\,. \tag{133}$$

$$m[Y_0^*(1405)]=m_\varphi+\tfrac{1}{2}m_\omega\,. \tag{134}$$

In connection with (132), we note that $m[K^*(1175)]=\frac{3}{2}m_\omega$, as will be discussed in Part (F). It may be noted that the (p,q) assignment for m_N-m_η is $(-4,4)$, i.e., $m_N-m_\eta=4\kappa-4m_\pi$. Thus the (p,q) doublet for m_N-m_η is somewhat similar to that for the ABC particle, namely $(-3,3)$. In both cases, we have $q=-p$. It follows from these assignments that

$$m_N-m_\eta=\tfrac{4}{3}m_{\text{ABC}}\,. \tag{135}$$

We have investigated the possibility that the eight pairs of particles involved in Eqs. (131) and (132) may have a common feature, in terms of the quantum numbers I, B, and S. We have found that in all eight cases, we have the relation

$$2\Delta I=\Delta B+\Delta S\,, \tag{136}$$

where ΔI, ΔB, and ΔS are the change of I, B, and S, respectively, in going from the state with higher mass to that with lower mass. We note that $\Delta B+\Delta S$ could also be written as ΔY, where Y is the hypercharge.

In applying Eq. (136), we must take the state with $S=+1$ for the K, $K^*(725)$, $K^*(888)$, and $K^*(1175)$ particles. For the baryons, we always use the particle and not the antiparticle state (i.e., we use $B=+1$). For the μ meson, we must take $I=0$. As an example, for $m_\Lambda-m[K^*(725)]$, we have $\Delta I=+\frac{1}{2}$, $\Delta B=-1$, $\Delta S=+2$, so that both sides of (136) equal $+1$. For $m[Y_0^*(1405)]-m_\varphi$, we have $\Delta I=0$, $\Delta B=-1$, $\Delta S=+1$. In connection with Eq. (132), it is assumed that the isotopic spin of $K^*(1175)$ is $I=\frac{1}{2}$, as reported in Ref. 24.

Equation (136) leads us to consider the following derived quantum number:

$$F\equiv I-\tfrac{1}{2}(B+S)\,. \tag{137}$$

The relation (136) obviously implies that $F_1=F_2$ for the two states belonging to each pair. For all of the presently known strongly interacting particles, one finds that F is either 0 or 1.

F. The Mass Relations $m_b=\lambda m_a$

It has been found that for several pairs of strongly interacting particles, we have the relation

$$m_b=\lambda m_a\,, \tag{138}$$

where λ is a simple fraction. We will first consider the cases where λ is half-integral (i.e., $\lambda=\frac{3}{2}$, $\frac{5}{2}$, or $\frac{7}{2}$), and quarter-integral (i.e., $\lambda=3/4$, $5/4$, $7/4$, $9/4$, or $11/4$). Later, we will consider cases where λ is equal to some other simple fractions.

The cases with $\lambda=\frac{1}{2}(2n+1)$ and $\lambda=\frac{1}{4}(2n+1)$ are listed in Tables IV and V, respectively. In each table, the last column gives the calculated value of m_b, as obtained from $m_b=\lambda m_a$. It is seen that the agreement with the experimental value[8] of m_b is in all cases to $\leqq 4$ MeV, and usually to $\leqq 2$ MeV. In this comparison, we have not included the uncertainties of the experimental values of m_a and m_b. In all cases, the agreement is within the limits of experimental errors for particles a and b.

In some of the cases, the relation (138) follows directly from the (p,q) assignments. As an example, the

TABLE IV. Pairs of particle states related by the equation: $m_b = \lambda m_a$, where $\lambda = \frac{3}{2}, \frac{5}{2}$, or $\frac{7}{2}$. The calculated mass of particle b is given in the last column of the table. (All values are in MeV.)

λ	Particle a (m_{exp})	Particle b (m_{exp})	$m_{b,\text{calc}}$
$\frac{3}{2}$	$\omega(782)$	$K^*(1175)$	1173
$\frac{3}{2}$	$N(939)$	$Y_0^*(1405)$	1408
$\frac{3}{2}$	$\varphi(1019)$	$\Xi_{1/2}^*(1532)$	1529
$\frac{5}{2}$	ABC(290)	$K^*(725)$	725
$\frac{5}{2}$	$K(496)$	$N_{3/2}^*(1238)$	1240
$\frac{5}{2}$	$K^*(725)$	$Y_0^*(1815)$	1813
$\frac{7}{2}$	ABC(290)	$\varphi(1019)$	1015
$\frac{7}{2}$	$\eta(549)$	$N_{3/2}^*(1922)$	1922

relation

$$m[N_{3/2}^*(1922)] = (7/2)m_\eta, \tag{139}$$

(cf. Table IV) is a consequence of Eq. (4), and $m_\eta = 4m_\pi$. In a similar manner the relation

$$m_\Lambda = \tfrac{3}{4}m[N_{1/2}^*(1485)] \tag{140}$$

follows from the (p,q) doublets, namely (3,3) for Λ and (4,4) for $N_{1/2}^*$.

Tables IV and V show that there exist several triplets of particles a, b, c, such that

$$m_c/m_b = m_b/m_a = \lambda. \tag{141}$$

Thus [ABC, $K^*(725)$, $Y_0^*(1815)$] form a triplet with $\lambda = 5/2$, and [ρ, N, $K^*(1175)$] and [η, $X(960)$, Ω] represent triplets with $\lambda = 5/4$ and $7/4$, respectively. There are altogether at least 9 such triplets among the observed particle states. A list of the triplets is given in Table VI.

It may be noted that if in Eq. (141) one considers the corresponding values of ln$m$ (to be denoted by w), one obtains

$$w_c - w_b = w_b - w_a = \ln\lambda. \tag{142}$$

Equation (142) is of the same form as that which defines a sequence of 3 particles in the mass spectrum ($m_3 - m_2 = m_2 - m_1$).

For $\lambda = 9/4$, the mass relations are of particular interest:

$$m_\Lambda = (9/4)m_K, \tag{143}$$

$$m[N_{3/2}^*(1238)] = (9/4)m_\eta, \tag{144}$$

$$m[N_{1/2}^*(1688)] = (9/4)m_\rho. \tag{145}$$

In this case, only Eq. (144) follows directly from the (p,q) values. The relation (143) has already been discussed above [see Eq. (95)]. It was shown that (143) is satisfied well within the limits of error of m_K. Equation (145) is also satisfied very accurately. It has been calculated that with a random distribution of masses, the probability for the agreement obtained with $\lambda = 9/4$ in Eqs. (143)–(145) is ~ 0.02.

We will now present some additional mass relations of the form $m_b = \lambda m_a$, where the denominator of λ is larger than 4. These relations may seem somewhat arbitrary at first. However, many of them follow from our previous results (Tables IV and V), or from the (p,q) assignments. Moreover, after most of this work was completed, it was realized that a few of these mass relations could also be derived from Takabayasi's results.[9]

These additional mass relations are listed in Table VII. Results are given for $\lambda = 6/5$, $7/5$, $8/5$, $9/5$; $13/10$; $10/9$, $20/9$; and $9/8$. The four cases with $\lambda = 13/10$ and the equation $m_B/m_\eta = 20/9$ are of some interest, since they may indicate that the relations (101) and (102) involving the μ meson are not accidental.

We note that although the relation (cf. Table VII):

$$m_\varphi = (13/10)m_\omega \tag{146}$$

was found independently, it could have been derived from the results of Takabayasi.[9] In Table I of his paper, it is shown that $m_\omega = 10\kappa_0$ and $m_\varphi = 13\kappa_0$, where $\kappa_0 \equiv \kappa/3 = 78.2$ MeV. Thus it follows that $m_\varphi/m_\omega = 13/10$. Similarly, the relation

$$m_f = (8/5)m_\omega \tag{147}$$

could also have been derived from Ref. 9, since $m_f = 16\kappa_0$, so that $m_f/m_\omega = 8/5$.

It is of interest to determine how many particle masses are related to the mass of a given particle by means of the relations of Tables IV–VII. As an example, one finds for the η meson, eight derived particle masses, and for the ω and N, five and seven derived mass values, respectively. Thus for the ω, the derived masses are those of the nucleon ($\lambda = 6/5$), φ($\lambda = 13/10$), $K^*(1175)$ ($\lambda = 3/2$), f($\lambda = 8/5$), and $Y_0^*(1405)$ ($\lambda = 9/5$). It should be also pointed out that the cases of Tables IV–VII constitute only a part (perhaps $\sim 1/2$) of the total number of mass relations where the denominator D_λ of λ is suitably small ($D_\lambda \leqq 10$). In any case, the mass relations given here establish already a large number of

TABLE V. Pairs of particle states related by the equation: $m_b = \lambda m_a$, where $\lambda = 3/4$, $5/4$, $7/4$, $9/4$, or $11/4$. The calculated mass of particle b is given in the last column of the table. (All values are in MeV.)

λ	Particle a (m_{exp})	Particle b (m_{exp})	$m_{b,\text{calc}}$
3/4	$f(1255)$	$N(939)$	941
3/4	$N_{1/2}^*(1485)$	$\Lambda(1115)$	1114
3/4	$N_{3/2}^*(1625)$	$B(1220)$	1224
5/4	$\rho(750)$	$N(939)$	938
5/4	$N(939)$	$K^*(1175)$	1174
5/4	$N_{1/2}^*(1688)$	$9_K(2112)$	2110
7/4	$\eta(549)$	$X(960)$	961
7/4	$X(960)$	$\Omega(1679)$	1680
7/4	$f(1255)$	$N_{1/2}^*(2196)$	2196
9/4	$K(496)$	$\Lambda(1115)$	1116
9/4	$\eta(549)$	$N_{3/2}^*(1238)$	1235
9/4	$\rho(750)$	$N_{1/2}^*(1688)$	1688
11/4	$\eta(549)$	$N_{1/2}^*(1512)$	1510

TABLE VI. Triplets of particle states (a,b,c) whose masses are such that $m_c=\lambda m_b=\lambda^2 m_a$.

λ	Particle a	Particle b	Particle c
5/2	ABC	$K^*(725)$	$Y_0^*(1815)$
5/4	ρ	N	$K^*(1175)$
7/4	η	$X(960)$	Ω
6/5	$K^*(1175)$	$Y_0^*(1405)$	$N_{1/2}^*(1688)$
9/8	Λ	f	$Y_0^*(1405)$
9/8	$K^*(1175)$	Ξ	$N_{1/2}^*(1485)$
4	π	η	$N_{1/2}^*(2197)$
4/3	$N_{3/2}^*(1238)$	$N_{1/2}^*(1647)$	$N_{1/2}^*(2197)$
2	η	$A_1(1090)$	$N_{1/2}^*(2197)$

connections between the different particle states. As an example, given the values of m_η, m_ω, and m_N, the present mass relations enable one to obtain the masses of the following 14 additional particles: π, K, ρ, $X(960)$, φ, $K^*(1175)$, B, $N_{3/2}^*(1238)$, f, $Y_0^*(1405)$, $N_{1/2}^*(1512)$, $N_{1/2}^*(1688)$, $N_{3/2}^*(1922)$, and $N_{1/2}^*(2197)$.

We could write down many additional mass relations of the type $m_b=\lambda m_a$, if the denominator of λ is made larger. Here we have restricted ourselves to the simplest of such relations, and we have found that they could be derived in several alternate ways in many cases.

IV. CONCLUSIONS

In this section, we will attempt to summarize the results which have been obtained, and their possible implications.

(1) The mass relations

$$m=pm_\pi+q\kappa \tag{42}$$

follow in a natural manner from some empirical observations on the mass spectrum, in particular, the fact that there are many cases of equal mass differences for different pairs of particles, and the related existence of sequences of particles with the same mass spacing Δm. In a general way, Eq. (42) demonstrates the underlying quantized nature of the mass spectrum. Moreover, the relations of Eq. (42) show that the mass, and in particular, the coefficients p and q cannot be completely determined from a knowledge of the other particle properties, i.e., from the values of the quantum numbers I, J, B, and S. The existence of the various sequences shows the interrelations of the mesons and the baryon states, i.e., of bosons and fermions. These interrelations are essentially empirical, and they do not seem to have been predicted by the present particle theories.

(2) The mass relation

$$m_3=m_1+m_2 \tag{76}$$

has been shown to exist in numerous cases. This relation has the same form as that for a particle production threshold, or for a composite particle with very small binding energy (quasinucleus). However, in the majority of the cases, the quantum numbers I, J, B, and S of particles 1 and 2 do not have the appropriate relations for the compound of particles 1 and 2 to have the same quantum numbers as particle 3. Thus the relations (76) must be regarded as mainly empirical.

As mentioned above, it appears that the coefficients p and q of Eq. (42) do not have a direct connection with the quantum numbers I, J, B, and S. Aside from several other similar situations, this result is derived from the two cases in each of which two particle states with the same quantum numbers I, J, B, S, and parity P have different mass values. Thus p and q cannot be unique functions of I, J, B, and S. The relation between (p,q) and (I,J,B,S) seems to be at a deeper level, in the following sense. The underlying theory of the strong interactions might an extension of the present SU_3 scheme.[20] Such a theory would probably determine which representations of the SU_3 group correspond to observed particles, e.g., whether the 27-fold representation exists in nature. From this theory, one might expect to predict on the one hand, the values of p and q, and on the other hand, the values of I, J, B, S, and parity of each particle.

(3) It has been found that the mass of the μ meson enters into several additive relations: $m_3=m_1+m_2$, where the other two particles, e.g., 1 and 3, are strongly interacting. To the extent that a theory could explain the additive relations (76), it therefore seems that, in some sense, the muon derives its mass from the strongly

TABLE VII. Pairs of particle states related by the equation: $m_b=\lambda m_a$, where $\lambda=6/5$, 7/5, 8/5, 9/5; 13/10; 10/9; 20/9; and 9/8. The calculated mass of particle b is given in the last column of the table. (All values are in MeV.)

λ	Particle a (m_{exp})	Particle b (m_{exp})	$m_{b,\text{calc}}$
6/5	$\omega(782)$	$N(939)$	938
6/5	$\varphi(1019)$	$B(1220)$	1223
6/5	$K^*(1175)$	$Y_0^*(1405)$	1410
6/5	$N_{3/2}^*(1238)$	$N_{1/2}^*(1485)$	1486
6/5	$Y_1^*(1385)$	$Y_1^*(1660)$	1662
6/5	$Y_0^*(1405)$	$N_{1/2}^*(1688)$	1686
6/5	$N_{1/2}^*(1512)$	$Y_0^*(1815)$	1814
7/5	$K^*(725)$	$\varphi(1019)$	1015
8/5	$\omega(782)$	$f(1255)$	1251
8/5	$\varphi(1019)$	$N_{3/2}^*(1625)$	1630
8/5	$N_{1/2}^*(1485)$	$N_{3/2}^*(2375)$	2376
9/5	$\omega(782)$	$Y_0^*(1405)$	1408
9/5	$N(939)$	$N_{1/2}^*(1688)$	1690
9/5	$B(1220)$	$N_{1/2}^*(2197)$	2196
9/5	$\Xi(1319)$	$N_{3/2}^*(2375)$	2374
13/10	$\omega(782)$	$\varphi(1019)$	1017
13/10	$N(939)$	$B(1220)$	1221
13/10	$f(1255)$	$N_{3/2}^*(1625)$	1631
13/10	$N_{1/2}^*(1688)$	$N_{1/2}^*(2197)$	2194
10/9	$K(496)$	$\eta(549)$	551
10/9	$\Lambda(1115)$	$N_{3/2}^*(1238)$	1239
10/9	$N_{1/2}^*(1512)$	$\Omega(1679)$	1680
10/9	$Y_0^*(1520)$	$N_{1/2}^*(1688)$	1689
20/9	$\eta(549)$	$B(1220)$	1220
9/8	$\Lambda(1115)$	$f(1255)$	1254
9/8	$f(1255)$	$Y_0^*(1405)$	1412
9/8	$K^*(1175)$	$\Xi(1319)$	1322
9/8	$\Xi(1319)$	$N_{1/2}^*(1485)$	1484

interacting particles. This possibility is made plausible by the fact that m_μ is much closer in order of magnitude to m_π than to m_e.

(4) The mass relations

$$m_b=\lambda m_a \quad (138)$$

seem to be in accordance with the expectation that all strongly interacting particles are, in some sense, equally fundamental. We note that Eq. (138) is symmetrical in m_a and m_b, since we can equally well write: $m_a=(1/\lambda)m_b$. In connection with the discussion following Eq. (147), which involves the η, ω, $N+14$ other particles, we could consider any of the 14 particles, e.g., $N_{1/2}^*(1688)$, and express the other 16 mass values in terms of $m[N_{1/2}^*(1688)]$. Thus, if we would assume that the 16 mass relations $m_b=\lambda m_a$ are exactly valid, irrespective of the values of m_a and m_b, then if we would make the mass of any one particle equal to 0, the masses of all other 16 particles would also vanish. In this sense, the relations (138) seem to indicate that the existence of any one particle and its mass implies the existence and mass of a large number of other particles.

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APPENDIX

In connection with the validity of Eq. (42), i.e., $m=pm_\pi+q\kappa$, we have made a statistical study in order to determine the probability P that a random distribution of masses would give the same agreement with Eq. (42) as is observed. As would be expected, this probability is negligibly small ($\sim 6\times10^{-4}$).

We note that there are 32 strongly interacting particle states which are known at present.[39] This number includes the three recently discovered resonances: $N_{1/2}^*(1485)$, $X(960)$, and $K^*(1175)$. In our comparison with a random distribution, we must exclude 5 states, namely π, N, $N_{3/2}^*(1625)$, $N_{1/2}^*(2190)$, and $N_{3/2}^*(2360)$. For π and N, the masses must be excluded from the comparison, since m_π and $\kappa=m_N/4$ are directly used in the mass equation. Moreover, the experimental mass values of $N_{3/2}^*(1625)$ [shoulder of the π^+p cross section], $N_{1/2}^*(2190)$, and $N_{3/2}^*(2360)$ are not known with sufficient accuracy to make a precise comparison with the values given by Eq. (42).

An inspection of the experimental and calculated mass values (Tables I–III) shows that the remaining 27 particle states can be divided into two groups, as follows: (1) those for which the difference $\delta m\equiv|m_{\text{exp}}-m_{\text{calc}}|$ is less than 4 MeV (20 states); (2) those for which δm exceeds 4 MeV (7 states). The maximum $\delta m=7.9$ MeV is obtained for $Y_0^*(1520)$. The 20 states in category (1) are the following: η, ω, φ, f, ABC, $X(960)$; K, $K^*(725)$, $K^*(1175)$; $N_{3/2}^*(1238)$, $N_{1/2}^*(1512)$, $N_{1/2}^*(1688)$, $N_{3/2}^*(1922)$; $N_{1/2}^*(1485)$; Λ, Σ, Ω, $Y_0^*(1405)$, Ξ, and $\Xi_{1/2}^*(1532)$. The seven states in category (2) are: ρ, $K^*(888)$, B, $Y_1^*(1385)$, $Y_0^*(1520)$, $Y_1^*(1660)$, and $Y_0^*(1815)$. In the comparison of m_{calc} with m_{exp}, we have used Rosenfeld's experimental value[8] whenever it is available, and we have disregarded the uncertainties of m_{exp}.

We have calculated all mass values of Eq. (42), for which p and q are integers subject to the condition $-3\leqq p\leqq+16$, $-2\leqq q\leqq+6$, and for which m lies between 0 and 2500 MeV. There are altogether 136 mass values in the region determined by the restrictions given above.

The region of $2\delta m=8$ MeV around each of the 136 calculated values would subtend a total region of $136\times8=1088$ MeV if there were no overlap between the 8-MeV bands pertaining to different (p,q) values. As a result of the overlap, the total subtended region is decreased from 1088 MeV to 1035.2 MeV. Since the total mass interval considered is 2500 MeV, the subtended region represents a fraction $1035.2/2500=0.4141$ of the total mass interval. It has been pointed out by Robinson[40] that if the allowed region constitutes a fraction f of the total mass region, then with 27 randomly distributed masses, the probability that at least 20 of them will fall into the allowed region ($\delta m\leqq4$ MeV), and seven may lie anywhere in the total mass interval is given by

$$P=\sum_{n=20}^{27}\frac{27!}{(27-n)!n!}f^n(1-f)^{27-n}. \quad (A1)$$

This equation can be derived by noting that the *a priori* probability that a particular set of n mass values will fall in the allowed region, and that the remaining $27-n$ values lie outside this region is simply given by $f^n(1-f)^{27-n}$. We must multiply this probability by the number of ways of choosing n objects out of a total of 27; i.e., by $27!/[(27-n)!n!]$. Finally, we must sum over all values of n between 20 and 27, to include the cases in which more than 20 mass values are in the allowed region.

It is found that the first term in the sum (A1), i.e., the term with $n=20$ predominates. With $f=0.4141$, we obtain

$$P=\frac{27!}{20!7!}(0.4141)^{20}(0.5859)^7\left[1+\left(\frac{7}{21}\right)\left(\frac{0.4141}{0.5859}\right)+\left(\frac{7}{21}\right)\left(\frac{6}{22}\right)\left(\frac{0.4141}{0.5859}\right)^2+\cdots\right]. \quad (A2)$$

[39] This number does not include the recently reported resonances σ, $A_1(1090)$, $A_2(1310)$, $N_{1/2}^*(1647)$, and $Y^*(1760)$, which were included in Fig. 1 after the main part of this paper had been completed. The inclusion of these states in the statistical study would not materially affect the final result for the probability P [Eq. (A2)].

[40] D. K. Robinson (private communication).

The square bracket of Eq. (A2) has the value 1.289. The factor preceding the square bracket is 4.624×10^{-4}, thus giving $P=5.96\times10^{-4}$ (3.4 standard deviations). Hence the probability that a random distribution of masses would give rise to the observed agreement with Eq. (42) is negligibly small, as was expected.

We have also made a calculation of the probability of obtaining the results concerning the mass differences $\Delta m\cong m_\mu$ and $\Delta m\cong 390$ MeV, if the mass distribution were random. For this purpose, we first calculate the probability of a given mass difference Δm on the assumption that the masses are randomly distributed between 0 and 2500 MeV. We let $x\equiv\Delta m/(2500\text{ MeV})$. For a mass difference Δm, the smaller mass can take on any value between 0 and $2500-\Delta m$, since the larger mass cannot exceed 2500 MeV, according to our assumption. Hence the relative probability of a given value of Δm is $2500-\Delta m$, or in terms of x, it is: $1-x$. In order to normalize to 1, we must take $P(x)=2(1-x)$, so that $\int_0^1 P(x)dx=1$.

For the muon mass, $m_\mu=105.6$ MeV, we have: $x=0.0422$, so that $P(x)=1.9156$. In the experimental mass spectrum we have five mass differences in a 2 MeV region extending from 105 to 107 MeV. This region, when weighted with the probability $P(x)$, corresponds to an effective fraction f_a:

$$f_a=\frac{2}{2500}\times1.9156=1.532\times10^{-3}. \tag{A3}$$

Similarly to Eq. (A1), the probability which we must calculate is given by

$$P_a=\sum_{n=5}^{n=496}\frac{496!}{(496-n)!n!}f_a{}^n(1-f_a)^{496-n}. \tag{A4}$$

Here 496 is the total number of mass differences among the 32 strongly interacting particle states considered here. As in the case of Eq. (A1), the term with lowest $n(=5)$ predominates. With $f_a=1.532\times10^{-3}$, the term $n=5$ equals 0.976×10^{-3}, and the total probability $P_a=1.108\times10^{-3}$, which would correspond to 3.3 standard deviations.

In a similar manner, one can calculate the probability that eight values of Δm would be closely equal to 390 MeV with a random mass distribution [cf. Sec. III, part (E)]. However, in this case, the situation is slightly more complicated. If one uses the best experimental values, and disregards the experimental errors, as has been done previously, then of the eight mass differences of Eqs. (131) and (132), only five have values in the region 390–392 MeV. These are $[K,\mu]$, $[K^*(888),K]$, $[\Lambda,K^*(725)]$, $[N,\eta]$, and $[N_{3/2}{}^*(1922),\Xi_{1/2}{}^*(1532)]$. For the remaining three pairs $[K^*(1175),\omega]$, $[Y_0{}^*(1405),\varphi]$, and $[N_{3/2}{}^*(1625),N_{3/2}{}^*(1238)]$, the mass differences are 393, 386, and 387 MeV, respectively. Of course, it should again be pointed out that the masses of some of these states are not known to a very high accuracy, i.e., to 2 MeV, in particular $K^*(1175)$ and $N_{3/2}{}^*(1625)$. Nevertheless, in our calculation of the probability, to be denoted by P_b, we will use only the five pairs with $\Delta m=390$ or 392 MeV. For the center of the region, $\Delta m=391$ MeV, we have: $x=391/2500=0.1564$, whence $P(x)=2(1-0.1564)=1.6872$. The effective fraction f_b for the 2-MeV region is therefore

$$f_b=\frac{2}{2500}\times1.6872=1.350\times10^{-3}. \tag{A5}$$

Since the mass difference 390 MeV does not correspond to the mass of an observed particle, we must calculate the probability that four pairs will have a mass difference which is equal to a predetermined value, i.e., the mass difference of the fifth pair (e.g., m_N-m_η). Hence the probability P_b is given by

$$P_b=\sum_{n=4}^{n=496}\frac{496!}{(496-n)!n!}f_b{}^n(1-f_b)^{496-n}. \tag{A6}$$

The term with $n=4$ predominates, and gives 4.26×10^{-3}. For the total probability P_b, one obtains 4.91×10^{-3}, which would correspond to 2.8 standard deviations. Both the probabilities P_a and P_b are very small, so that the equality of the mass differences both for m_μ and for 390 MeV$=4\kappa-4m_\pi$ is probably physically significant.

It may be remarked that if we had seven pairs in the interval from 390 to 392 MeV (instead of five), the probability P_b would be decreased to 6.93×10^{-5} (4.0 standard deviations).